hep-th/yymm.nnnn

Janus Configurations, Chern-Simons Couplings, And The θ-Angle in $\mathcal{N} = 4$ Super Yang-Mills Theory

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Abstract

We generalize the half-BPS Janus configuration of four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory to allow the theta-angle, as well as the gauge coupling, to vary with position. We show that the existence of this generalization is closely related to the existence of novel three-dimensional Chern-Simons theories with $\mathcal{N} = 4$ supersymmetry. Another closely related problem, which we also elucidate, is the D3-NS5 system in the presence of a four-dimensional theta-angle.

Contents

1 Introduction

In this paper, we will consider, in four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory, several questions that involve θ-dependence, different choices of unbroken supersymmetry algebra, and relations to three-dimensional supersymmetric theories with Chern-Simons interactions.

We start by reconsidering the Janus solution [1, 2]. In $\mathcal{N} = 4$ super Yang-Mills theory, the Janus solution corresponds to a situation in which the coupling constant depends non-trivially on one of the spatial coordinates, which we will call y. The original Janus solution preserved the full R-symmetry group of the theory and violated all supersymmetry. Later,

however, variants were found that preserve some supersymmetry and only part of the R-symmetry [3], and in fact it is possible to preserve one-half of the full supersymmetry [4]. The unbroken supersymmetry algebra is then $OSp(4|4)$, which is a "half-BPS" subalgebra of the full $PSU(4|4)$ symmetry algebra of $\mathcal{N}=4$ super Yang-Mills theory. This is the case that we focus on.

Originally, Janus solutions were found in supergravity, but counterparts, which we will call Janus configurations, also exist [2,5] in weakly coupled field theory. However, the known field theory constructions are not as general as what has been found in supergravity. The known field theory constructions are limited to the case that only the gauge coupling g, and not the theta-angle θ, depends non-trivially on y.

One goal of this paper is to generalize the Janus solution to the case that both θ and g depend on y. This is accomplished in section 2. As we explain there, a key input is the fact that the relevant unbroken supersymmetry algebra has inequivalent embeddings in $PSU(4|4)$. To make θ become y-dependent, one must use a y-dependent embedding of the superalgebra.

The problem of making θ to be y-dependent is related to the problem of Chern-Simons couplings in three-dimensional gauge theory. Let us consider a four-dimensional gauge theory with θ a function of y, which is one of the four coordinates. The relevant part of the action is

$$I_\theta = -\frac{1}{32\pi^2}\int \mathrm{d}^4x\,\theta(y)\,\epsilon^{\mu\nu\alpha\beta}\,\mathrm{Tr}\,F_{\mu\nu}F_{\alpha\beta}. \tag{1.1}$$

We write $\mathrm{d}^4x = \mathrm{d}^3x\,\mathrm{d}y$. After integrating by parts and dropping any surface terms, I_θ is equivalent to

$$I_\theta = \frac{1}{8\pi^2}\int \mathrm{d}^3x\,dy\,\frac{\mathrm{d}\theta}{\mathrm{d}y}\epsilon^{\mu\nu\lambda}\mathrm{Tr}\,\left(A_\mu\partial_\nu A_\lambda + \frac{2}{3}A_\mu A_\nu A_\lambda\right). \tag{1.2}$$

This interaction is similar to a three-dimensional Chern-Simons interaction, so supersymmetrizing a four-dimensional theory with a y-dependent θ angle is somewhat similar to supersymmetrizing a three-dimensional theory with a Chern-Simons interaction.

We therefore re-examine the problem of supersymmetrizing the three-dimensional Chern-Simons coupling. Quite a few results are already known. $\mathcal{N}=3$ supersymmetry (in the three-dimensional sense) allows one to add a Chern-Simons coupling to a general three-dimensional gauge theory that also has the conventional F^2 kinetic energy [6–9]. It has been argued [10] that there are additional possibilities if one omits the usual kinetic energy, and recently a Chern-Simons theory with $\mathcal{N}=8$ supersymmetry (and no F^2 term) was constructed [11]. This construction was very special: the gauge group is $SO(4)$, and the matter representation is uniquely determined.

In our problem, the unbroken supersymmetry corresponds to $\mathcal{N}=4$ in the three-

dimensional sense, so in section 3.2, we consider three-dimensional Chern-Simons theories with this amount of supersymmetry, and no F^2 coupling. Our approach is to assume $\mathcal{N}=1$ supersymmetry, which admits a convenient superspace description, and then restrict the couplings so that a global $SO(4)$ symmetry appears, promoting $\mathcal{N}=1$ to $\mathcal{N}=4$. Moreover, we take the superpotential to be quartic so that, just as in [10,11], the theories we construct are conformally invariant at the classical level, and presumably also quantum mechanically.

We are able to completely classify theories of this kind, in terms of supergroups. The gauge group is the bosonic part of a supergroup, and the matter representation is determined by the fermionic part of that supergroup. Leaving aside theories with abelian gauge symmetry or associated with certain exceptional supergroups, the main examples correspond to the supergroups $U(N|M)$ (or their cousins $SU(N|M)$ and $PSU(N|N)$) and $OSp(N|M)$. The gauge groups are $U(N)\times U(M)$ and $O(N)\times Sp(M)$, and the matter fields are in the bifundamental representations.

The same groups and representations arise in the theory of D3-branes interacting with NS5-branes. This fact suggests that it would be fruitful to combine the following three problems: Janus configurations, D3-branes ending on fivebranes, and three-dimensional Chern-Simons couplings. This is our goal in the rest of the paper. In section 3.4, we repeat the analysis of section 2 using three-dimensional $\mathcal{N}=1$ superfields. In contrast to section 2, in which we start with the full R-symmetry and constrain the couplings to get supersymmetry, here we start with $\mathcal{N}=1$ supersymmetry (in the three-dimensional sense) and constrain the couplings to get the full R-symmetry. The two approaches lead to the same Lagrangians with the same supersymmetry.

In section 3.5, we apply this method to the D3-NS5 system. We find a close parallel with the purely three-dimensional results of section 3.2, and this enables us to resolve a riddle. This system is usually considered at $\theta=0$, and its appropriate description for $\theta\neq 0$ does not seem to be known. The answer is given by a special case of our construction. Equivalently, we can use our method to describe at $\theta=0$ a system consisting of a D3-brane and a $(1,q)$ five-brane (a combination of an NS-fivebrane and q D-fivebranes). The low energy description of this system has also not been understood in the literature. Closing the circle, we show that the Janus configuration can be recovered from a knowledge of the D3-NS5 system with general couplings.

2 Janus Configuration With Spatially Varying Theta Angle

2.1 Preliminaries

$\mathcal{N}=4$ super Yang-Mills theory is conveniently obtained by dimensional reduction from ten dimensions [12]. We begin in $\mathbb{R}^{1,9}$, with metric g_{IJ}, $I,J=0,\ldots,9$ of signature $-++\cdots+$. Gamma matrices Γ_I obey $\{\Gamma_I,\Gamma_J\}=2g_{IJ}$, and the supersymmetry generator is a Majorana-Weyl spinor ε, obeying $\overline{\Gamma}\varepsilon=\varepsilon$, where $\overline{\Gamma}=\Gamma_0\Gamma_1\cdots\Gamma_9$. The fields are a gauge field A_I and Majorana-Weyl fermion Ψ, also obeying $\overline{\Gamma}\Psi=\Psi$. Thus, ε and Ψ both transform in the **16** of $SO(1,9)$. The supersymmetric action is

$$I=\frac{1}{e^2}\int \mathrm{d}^{10}x\,\mathrm{Tr}\,\left(\frac{1}{2}F_{IJ}F^{IJ}-i\overline{\Psi}\Gamma^I D_I\Psi\right). \tag{2.1}$$

The conserved supercurrent is

$$J^I=\frac{1}{2}\mathrm{Tr}\,\Gamma^{JK}F_{JK}\Gamma^I\Psi, \tag{2.2}$$

and the supersymmetry transformations are

$$\delta A_I=i\overline{\varepsilon}\Gamma_I\Psi \tag{2.3}$$

$$\delta\Psi=\frac{1}{2}\Gamma^{IJ}F_{IJ}\varepsilon. \tag{2.4}$$

We reduce to four dimensions by simply declaring that the fields are allowed to depend only on the first four coordinates $x^0,\ldots,x^3$. This breaks the ten-dimensional Lorentz group $SO(1,9)$ to $SO(1,3)\times SO(6)_R$, where $SO(1,3)$ is the four-dimensional Lorentz group and $SO(6)_R$ is a group of R-symmetries. Actually, the fermions transform as spinors of $SO(6)_R$, and the R-symmetry group of the full theory is really $Spin(6)_R$, which is the same as $SU(4)_R$. The ten-dimensional gauge field splits as a four-dimensional gauge field A_μ, $\mu=0,\ldots,3$, and six scalars fields A_{3+i}, $i=1,\ldots,6$ that we rename as Φ_i. They transform in the fundamental representation of $SO(6)_R$. The supersymmetries ε and fermions Ψ transform under $SO(1,3)\times SO(6)_R$ as $(\mathbf{2},\mathbf{1},\mathbf{4})\oplus(\mathbf{1},\mathbf{2},\overline{\mathbf{4}})$, where $(\mathbf{2},\mathbf{1})$ and $(\mathbf{1},\mathbf{2})$ are the two complex conjugate spinor representations of $SO(1,3)$ and $\mathbf{4}$, $\overline{\mathbf{4}}$ are the two complex conjugate spinor representations of $SO(6)_R$.

In a Janus configuration, the coupling parameters depend non-trivially on one of the four spacetime coordinates, which we take to be $y=x^3$. To preserve half of the supersymmetry, it is necessary to break the R-symmetry from $SO(6)$ to $SO(3)\times SO(3)$. A special case of

a Janus configuration is one in which the couplings jump discontinuously at, say $y = 0$. Such a configuration is invariant under those conformal transformations that preserve the plane $y = 0$. The group of such conformal transformations is the three-dimensional conformal supergroup $SO(2,3)$, whose double cover is (the split real form) $Sp(4,\mathbb{R})$. The corresponding supergroup is $OSp(4|4)$, whose bosonic part is $SO(4) \times Sp(4)$. The second factor is the conformal supergroup and the first factor is the R-symmetry group ($SO(4)$ is a double cover of $SO(3) \times SO(3)$). The spatial variation of couplings in a conformally invariant Janus configuration reduces $PSU(4|2,2)$ to $OSp(4|4)$. A more general Janus configuration reduces $PSU(4|2,2)$ to the subalgebra of $OSp(4|4)$ consisting of symmetries that preserve a metric in spacetime. This subalgebra is usually called the three-dimensional global supersymmetry algebra with $\mathcal{N} = 4$ supersymmetry (8 supercharges) and R-symmetry group $SO(4)$.

2.1.1 Outer Automorphism

A key feature of this problem is that there is a one-parameter family of inequivalent embeddings of $OSp(4|4)$ in $PSU(4|2,2)$. The reason for this is that $PSU(4|2,2)$ has a one-parameter group of outer automorphisms. Represent an element M of $PSU(4|2,2)$ by a supermatrix

$$M = \begin{pmatrix} S & T \\ U & V \end{pmatrix} \tag{2.5}$$

where S and V are bosonic 4×4 blocks and U and T are fermionic ones (in $PSU(4|4)$, M has superdeterminant 1 and is identified with λM for any scalar λ). Then $PSU(4|4)$ has a group $F \cong U(1)$ of outer automorphisms, acting by $M \to VMV^{-1}$ with

$$V = \begin{pmatrix} e^{i\beta} & 0 \\ 0 & 1 \end{pmatrix}, \ \beta \in \mathbb{R}. \tag{2.6}$$

Conjugation by $U(1)$ generates a one-parameter family of embeddings of $OSp(4|4)$ in $PSU(4|4)$.

Concretely, the fermionic generators of $PSU(4|2,2)$ transform under the bosonic subgroup $SU(4) \times SU(2,2)$ as $\mathbf{4} \otimes \overline{\mathbf{4}}' \oplus \overline{\mathbf{4}} \otimes \mathbf{4}'$, where $\mathbf{4}$ and $\mathbf{4}'$ are the four-dimensional representations of $SU(4)$ and $SU(2,2)$, respectively. Once we reduce $SU(4)$ and $SU(2,2)$ to $SO(4)$ and $Sp(4,\mathbb{R})$, the representations $\mathbf{4}$ and $\mathbf{4}'$ become real. So, as a representation of $SO(4) \times Sp(4,\mathbb{R})$, the fermionic generators of $PSU(4|2,2)$ consist of two copies of the real representation $\mathbf{4} \otimes \mathbf{4}'$. These two copies are rotated by the outer automorphism group $F \cong SO(2)$. If we pick any linear combination of the two copies of $\mathbf{4} \otimes \mathbf{4}'$, then this, together with the Lie algebra of $SO(4) \times Sp(4,\mathbb{R})$, gives an $OSp(4|4)$ subalgebra of $PSU(4|4)$.

Though we have described this in the conformally-invariant case, conformal invariance is not essential. All statements remain valid if we replace $Sp(4,\mathbb{R})$ by its subgroup that

preserves a metric; this is simply the three-dimensional Poincaré group. In its action on the fermions, the Poincaré group reduces to the three-dimensional Lorentz group $SO(1,2)$.

Conformally-invariant Janus configurations are particularly interesting, but they are not generic. We will not impose conformal invariance in the following analysis.

2.1.2 Notation

It is convenient to split the scalars Φ_i, $i = 1, \ldots, 6$ into two groups acted on respectively by the two factors of $SO(3) \times SO(3) \subset SO(6)_R$. We take these two groups to consist of the first three and last three[1] Φ's; we rename (Φ_1, Φ_2, Φ_3) as $\vec{X} = (X_1, X_2, X_3)$ and (Φ_4, Φ_5, Φ_6) as $\vec{Y} = (Y_1, Y_2, Y_3)$. We sometimes write $SO(3)_X$ and $SO(3)_Y$ for the two $SO(3)$ groups.

Though the $\mathbf{16}$ of $SO(1,9)$, in which the supersymmetries transform, is irreducible, it is reducible as a representation of $W = SO(1,2) \times SO(3)_X \times SO(3)_Y$. Indeed, the action of W commutes with the three operators

$$\begin{aligned} B_0 &= \Gamma_{456789} \\ B_1 &= \Gamma_{3456} \\ B_2 &= \Gamma_{3789}. \end{aligned} \tag{2.7}$$

They obey $B_0^2 = -1$, $B_1^2 = B_2^2 = 1$, and $B_0 B_1 = -B_1 B_0 = B_2$, etc., and generate an action of $SL(2,\mathbb{R})$. We can decompose the $\mathbf{16}$ of $SO(1,9)$ as $V_8 \otimes V_2$, where V_8 transforms in the real irreducible representation $(\mathbf{2},\mathbf{2},\mathbf{2})$ of $SO(1,2) \times SO(3)_X \times SO(3)_Y$, and V_2 is a two-dimensional space in which the B_i are represented by

$$\begin{aligned} B_0 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ B_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ B_2 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \tag{2.8}$$

A W-invariant embedding of the eight supercharges of three-dimensional $\mathcal{N} = 4$ supersymmetry into the four-dimensional supersymmetry algebra can be obtained by putting a constraint on the supersymmetry generators

$$(\sin\psi B_1 + \cos\psi B_2)\,\varepsilon = \varepsilon, \tag{2.9}$$

[1]In ten-dimensional notation, $\vec{X}$ is related to x^4, x^5, x^6 and $\vec{Y}$ to x^7, x^8, x^9.

for some ψ. The possible choices are rotated by the one-parameter group generated by B_0. This is the outer automorphism group F. If ε and $\widetilde{\varepsilon}$ obey (2.9) (with the same value of ψ), then

$$\overline{\varepsilon}\Gamma_3\widetilde{\varepsilon} = 0. \tag{2.10}$$

The physical meaning of this is that, as a Janus configuration is not invariant under translations of $y = x^3$, the anticommutator of two fermionic symmetries of such a configuration never generates a translation in the y direction.

2.2 Construction

$\mathcal{N} = 4$ super Yang-Mills theory has been generalized in [5] to allow a y-dependent coupling constant while preserving half the supersymmetry. We will extend this to include a varying θ angle.

We begin with the unperturbed $\mathcal{N} = 4$ action

$$I = \int \mathrm{d}^4x \frac{1}{e^2} \operatorname{Tr} \left(\frac{1}{2} F_{IJ}F^{IJ} - i\overline{\Psi}\Gamma^I D_I \Psi \right) \tag{2.11}$$

and supersymmetry transformations

$$\delta A_I = i\overline{\varepsilon}\Gamma_I \Psi \tag{2.12}$$

$$\delta\Psi = \frac{1}{2}\Gamma^{IJ}F_{IJ}\varepsilon. \tag{2.13}$$

We will perturb both the action and the supersymmetry transformations to be y-dependent, while preserving half of the supersymmetry. The generators ε of the unbroken supersymmetries will themselves also be y-dependent. (This fact is perhaps the main novelty in our analysis here.) However, the y-dependence of ε is rather special. The commutators of two unbroken supersymmetries will be, of course, a translation in the directions x^0, x^1, x^2, with y-independent coefficients (since y-dependent translations of the other coordinates do not give symmetries). This is tantamount to the condition

$$\frac{\mathrm{d}}{\mathrm{d}y}\overline{\varepsilon}\Gamma^\mu\varepsilon = 0, \quad \mu = 0, 1, 2. \tag{2.14}$$

For this to hold, the y-dependence of ε must be generated by the outer automorphism group F. This result will emerge below from our explicit calculation (see eqn. (2.28)).

Now we describe the corrections to the supersymmetry transformations and to the action. Dimensional analysis permits us to add a correction to the supersymmetry transformation

of Ψ:

$$\widetilde{\delta}\Psi = \frac{1}{2}\left(\Gamma \cdot X\left(s_1\Gamma_{456} + s_2\Gamma_{789}\right) + \Gamma \cdot Y\left(t_1\Gamma_{456} + t_2\Gamma_{789}\right)\right)\varepsilon, \tag{2.15}$$

where $\Gamma{\cdot}X$ and $\Gamma{\cdot}Y$ are abbreviations, respectively, for $\sum_a \Gamma_a X^a$ and $\sum_p \Gamma_p Y^p$. Here s_1, s_2, t_1 and t_2 will be functions of $y = x^3$, along with other parameters that appear momentarily.

To the action, we can add fermion bilinear terms:

$$I' = \int \mathrm{d}^4x \frac{i}{e^2} \mathrm{Tr}\, \overline{\Psi}\left(\alpha\Gamma_{012} + \beta\Gamma_{456} + \gamma\Gamma_{789}\right)\Psi. \tag{2.16}$$

This is the most general fermion bilinear that is gauge-invariant and has $SO(1,2) \times SO(3) \times SO(3)$ symmetry. It is also possible to add the following dimension 3 bosonic terms to the action[2]:

$$I'' = \int \mathrm{d}^4x \frac{1}{e^2}\left(u\epsilon^{\mu\nu\lambda}\mathrm{Tr}\left(A_\mu\partial_\nu A_\lambda + \frac{2}{3}A_\mu A_\nu A_\lambda\right) + \frac{v}{3}\epsilon^{abc}\mathrm{Tr}\, X_a[X_b, X_c] + \frac{w}{3}\epsilon^{pqr}\mathrm{Tr}\, Y_p[Y_q, Y_r]\right). \tag{2.17}$$

Finally, the action can have terms of dimension 2:

$$I''' = \int d^4x \mathrm{Tr}\left(\frac{r}{2e^2}X_a X^a + \frac{\widetilde{r}}{2e^2}Y_p Y^p\right). \tag{2.18}$$

It is convenient to define

$$q = e^2 \frac{\mathrm{d}}{\mathrm{d}y}\frac{1}{e^2}. \tag{2.19}$$

We consider q and similarly $\mathrm{d}\varepsilon/\mathrm{d}y$ and the parameters $\alpha, \beta, \gamma,\ s_i, t_j$, and u, v, w to be of first order, while r and $\widetilde{r}$, the second derivatives of e^2 and ε, and the first derivatives of the other parameters are second order. (Of course, homogeneous quadratic expressions in first order quantities are second order also.) We already know that the zeroth order variation of I vanishes, since the pure $\mathcal{N} = 4$ theory in four dimensions is supersymmetric. We need to examine the supersymmetry of the first and second order quantities.

2.3 First Order Variations

When we act with the unperturbed supersymmetry variation δ on the unperturbed action I, the zeroth order terms vanish, as just noted, but we do get first order terms involving the

[2] The ϵ symbols that appear here are antisymmetric tensors in the 012, 456, and 789 subspaces, normalized to $\epsilon^{012} = \epsilon^{456} = \epsilon^{789} = 1$.

y derivatives of e^2 and ε:

$$\delta I|_1 = -i\,\mathrm{Tr}\int \mathrm{d}^4x \frac{1}{e^2}\frac{\mathrm{d}\overline{\varepsilon}}{\mathrm{d}y}\Gamma^{KL}F_{KL}\Gamma_3\Psi \tag{2.20}$$

$$-\frac{i}{2}\,\mathrm{Tr}\int \mathrm{d}^4x \frac{q}{e^2}\overline{\varepsilon}\Gamma_3\Gamma^{KL}F_{KL}\Psi. \tag{2.21}$$

First order variations come from several other places. The unperturbed supersymmetry variation δ acting on the perturbed action I' gives

$$\delta I' = -i\,\mathrm{Tr}\int \mathrm{d}^4x \frac{1}{e^2}\overline{\varepsilon}\Gamma^{IJ}F_{IJ}\left(\alpha\Gamma_{012}+\beta\Gamma_{456}+\gamma\Gamma_{789}\right)\Psi. \tag{2.22}$$

The perturbed supersymmetry variation $\widetilde{\delta}$ acting on the unperturbed action I gives a first order contribution

$$\widetilde{\delta} I|_1 = i\,\mathrm{Tr}\int d^4x \frac{1}{e^2}\overline{\varepsilon}\left((s_1\Gamma_{456}+s_2\Gamma_{789})\Gamma^{Ia}D_IX_a+(t_1\Gamma_{456}+t_2\Gamma_{789})\Gamma^{Ip}D_IY_p)\right)\Psi. \tag{2.23}$$

The remaining first order terms come from $\delta I''$:

$$\delta I'' = i\mathrm{Tr}\int d^4x \frac{1}{e^2}\left(u\,\epsilon^{\mu\nu\lambda}\overline{\varepsilon}\Gamma_\mu\Psi F_{\nu\lambda}+v\epsilon^{abc}\overline{\varepsilon}\Gamma_a\Psi[X_b,X_c]+w\epsilon^{pqr}\overline{\varepsilon}\Gamma_p\Psi[Y_q,Y_r]\right). \tag{2.24}$$

Now let us give some samples of the use of the above formulas. First we consider variations proportional to $D_\mu X_a\Psi$, contracted with $\overline{\varepsilon}$ and some gamma matrices. The sum of such contributions comes out to be

$$i\int \mathrm{d}^4x\frac{1}{e^2}\left(-2\frac{\mathrm{d}\overline{\varepsilon}}{\mathrm{d}y}-q\overline{\varepsilon}+2\overline{\varepsilon}(\alpha\Gamma_{0123}-\beta\Gamma_{3456}+\gamma\Gamma_{3789})-\overline{\varepsilon}\left(s_1\Gamma_{3456}+s_2\Gamma_{3789}\right)\right)\mathrm{Tr}\,D_\mu X_a\Gamma_{\mu a3}\Psi. \tag{2.25}$$

The condition for vanishing of terms of this form therefore gives

$$-2\frac{\mathrm{d}\overline{\varepsilon}}{\mathrm{d}y}-q\overline{\varepsilon}+2\overline{\varepsilon}(\alpha\Gamma_{0123}-\beta\Gamma_{3456}+\gamma\Gamma_{3789})-\overline{\varepsilon}\left(s_1\Gamma_{3456}+s_2\Gamma_{3789}\right)=0. \tag{2.26}$$

A similar analysis of terms proportional to $D_3X_a\Psi$ gives a very similar equation with a couple of signs reversed:

$$2\frac{\mathrm{d}\overline{\varepsilon}}{\mathrm{d}y}-q\overline{\varepsilon}+2\overline{\varepsilon}(-\alpha\Gamma_{0123}-\beta\Gamma_{3456}+\gamma\Gamma_{3789})-\overline{\varepsilon}\left(s_1\Gamma_{3456}+s_2\Gamma_{3789}\right)=0. \tag{2.27}$$

By subtracting the last two equations, we get an equation that determines the y-dependence of $\overline{\varepsilon}$:

$$\frac{\mathrm{d}\overline{\varepsilon}}{\mathrm{d}y}=\alpha\overline{\varepsilon}\Gamma_{0123}. \tag{2.28}$$

(This shows that ε varies with y by an element of the outer automorphism group F, for reasons explained at the beginning of section 2.2). We will also need the transpose

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}y} = \alpha\Gamma_{0123}\varepsilon. \tag{2.29}$$

Notice that as expected

$$\frac{\mathrm{d}}{\mathrm{d}y}\overline{\varepsilon}\Gamma^3\widetilde{\varepsilon} = \alpha\overline{\varepsilon}\Gamma_{0123}\Gamma^3\varepsilon + \alpha\overline{\varepsilon}\Gamma^3\Gamma_{0123}\varepsilon = 0 \tag{2.30}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}y}\overline{\varepsilon}\Gamma^\mu\widetilde{\varepsilon} = \alpha\overline{\varepsilon}\Gamma_{0123}\Gamma^\mu\varepsilon + \alpha\overline{\varepsilon}\Gamma^\mu\Gamma_{0123}\varepsilon = 0 \tag{2.31}$$

The sum of the two equations gives

$$0 = \overline{\varepsilon}\left((s_1 + 2\beta)\Gamma_{3456} + (s_2 - 2\gamma)\Gamma_{3789} + q\right). \tag{2.32}$$

It is convenient to regard this as an equation that uniquely determines s_1 and s_2 in terms of $\overline{\varepsilon}$, β, γ, and q. Indeed, if written out explicitly, eqn. (2.32) is equivalent to a pair of linear equations that have a unique solution for the unknowns s_1, s_2. Explicitly solving for s_1 and s_2 makes the formulas more complicated, and for now it is more convenient to simply leave the equation for s_1 and s_2 in the given form.

Upon exchanging X and Y, and considering terms proportional to $D_\mu Y_p\Psi$ or $D_3Y_p\Psi$, we get two more similar equations. One linear combination gives (2.27) again, and the second gives the counterpart of (2.32):

$$0 = \overline{\varepsilon}\left((t_1 - 2\beta)\Gamma_{3456} + (t_2 + 2\gamma)\Gamma_{3789} + q\right). \tag{2.33}$$

A similar analysis of terms proportional to $[X_a, Y_p]\Psi$ gives the following condition:

$$-2\frac{\mathrm{d}\overline{\varepsilon}}{\mathrm{d}y} - q\overline{\varepsilon} - 2\overline{\varepsilon}(\alpha\Gamma_{0123} + \beta\Gamma_{3456} + \gamma\Gamma_{3789}) - \overline{\varepsilon}((s_1+t_1)\Gamma_{3456} + (s_2+t_2)\Gamma_{3789}) = 0. \tag{2.34}$$

With the aid of the above formulas, this reduces to

$$0 = \overline{\varepsilon}\left(4\alpha\Gamma_{0123} + 2\beta\Gamma_{3456} + 2\gamma\Gamma_{3789} - q\right). \tag{2.35}$$

This equation determines β and γ in terms of ε, α, and q, and then eqns. (2.32) and (2.33) similarly determine s_1, s_2, t_1 and t_2 in terms of the same variables.

The analysis of the remaining first order terms in the variation of the action is similar. The terms proportional to $D_3X^a\Psi$ vanish with the aid of the above formulas. The vanishing

of terms proportional to $F_{\mu\nu}\Psi$, $[X_a, X_b]\Psi$, and $[Y_p, Y_q]\Psi$ serves, respectively, to determine the coefficients u, v, and w in eqn (2.17).

Let us work out the terms $F_{\mu\nu}\Psi$. In doing so, for brevity we omit the usual factors $i\int \mathrm{d}^4x\frac{1}{e^2}\,\mathrm{Tr}$, leaving the integration and the trace understood. From $\delta I|_1$, we get

$$-\overline{\varepsilon}\alpha\Gamma_{0123}\Gamma^{\mu\nu}F_{\mu\nu}\Gamma_3\Psi - \frac{1}{2}\overline{\varepsilon}q\Gamma_3\Gamma^{\mu\nu}F_{\mu\nu}\Psi. \tag{2.36}$$

And from $\delta I'$, we get

$$-\overline{\varepsilon}\Gamma^{\mu\nu}F_{\mu\nu}\left(\alpha\Gamma_{012} + \beta\Gamma_{456} + \gamma\Gamma_{789}\right)\Psi. \tag{2.37}$$

There is no contribution from $\widetilde{\delta} I|_1$. These contributions add to

$$-\overline{\varepsilon}\left(2\alpha\Gamma_{0123} - \beta\Gamma_{3456} - \gamma\Gamma_{3789} + \frac{q}{2}\right)\Gamma^{\mu\nu}F_{\mu\nu}\Gamma_3\Psi. \tag{2.38}$$

With the aid of eqn. (2.35), this collapses to

$$-4\alpha\overline{\varepsilon}\Gamma_{0123}\Gamma^{\mu\nu}F_{\mu\nu}\Gamma_3\Psi = 4\alpha\overline{\varepsilon}\epsilon^{\mu\nu\lambda}F_{\mu\nu}\Gamma_\lambda. \tag{2.39}$$

Comparing to (2.24), we see that the remaining supersymmetry variation $\delta I''$ will cancel this term precisely if

$$u = -4\alpha. \tag{2.40}$$

A very similar analysis of the terms $[X_a, X_b]\Psi$ and $[Y_p, Y_q]\Psi$ shows that these contributions to the supersymmetry variation similarly cancel if

$$v = -4\beta, \quad w = -4\gamma. \tag{2.41}$$

To summarize what we have obtained so far, we may begin with two arbitrary functions $\alpha(y)$ and $q(y)$ and an arbitrary initial value of $\overline{\varepsilon}(y)$ at, say, $y = y_0$. The y-dependence of $\overline{\varepsilon}$ is then determined from eqn. (2.28), and the other equations determine everything else in terms of α, q, and ε. So far $\alpha(y)$ and $q(y)$ are arbitrary, but it turns out that vanishing of the second order variations places a non-trivial restriction on these functions.

2.4 Second Order Variations

There are three sources of second order variations.

The supersymmetry variation of I''', the part of the action that is of dimension 2, is easily computed:

$$\delta I''' = i\int \mathrm{d}^4x\frac{1}{e^2}\mathrm{Tr}\,\overline{\varepsilon}\left(r\Gamma\cdot X + \widetilde{r}\Gamma\cdot Y\right)\Psi. \tag{2.42}$$

It is of second order simply because we consider r and $\widetilde{r}$ to be second order quantities.

The modified supersymmetry variation $\widetilde{\delta}$ acting on the correction I' to the action, is again not difficult to compute:

$$\widetilde{\delta}I' = -i\int \mathrm{d}^4x\frac{1}{e^2}\mathrm{Tr}\,\overline{\varepsilon}\Big(\big((s_1\Gamma_{456}+s_2\Gamma_{789})\Gamma\cdot X+(t_1\Gamma_{456}+t_2\Gamma_{789}\Gamma\cdot Y)\big)(\alpha\Gamma_{012}+\beta\Gamma_{456}+\gamma\Gamma_{789})\Big)\Psi. \tag{2.43}$$

This is equivalent to

$$\widetilde{\delta}I' = -i\int \mathrm{d}^4x\frac{1}{e^2}\mathrm{Tr}\,\overline{\varepsilon}\Big(\big((s_1\Gamma_{3456}+s_2\Gamma_{3789})\Gamma\cdot X+(t_1\Gamma_{3456}+t_2\Gamma_{3789}\Gamma\cdot Y)\big)(-\alpha\Gamma_{0123}+\beta\Gamma_{3456}+\gamma\Gamma_{3789})\Big)\Psi. \tag{2.44}$$

This can be further simplified using eqns. (2.32) and (2.33). The terms involving X become

$$\widetilde{\delta}I'_X = -i\int \mathrm{d}^4x\frac{1}{e^2}\mathrm{Tr}\,\overline{\varepsilon}\Big((2\beta^2+2\gamma^2+(2\gamma\alpha+q\beta)\Gamma_{3456}+(2\beta\alpha-q\gamma)\Gamma_{3789}+q\alpha\Gamma_{0123})\Gamma\cdot X\Psi\Big). \tag{2.45}$$

The terms involving Y can be analyzed similarly, but one can also take a short cut using symmetry, as we explain below.

The remaining second order terms are

$$\begin{aligned}\widetilde{\delta}I|_2 = &-i\int \mathrm{d}^4x\mathrm{Tr}\,\Bigg(\Big(\frac{q}{2e^2}+\frac{1}{e^2}\frac{\mathrm{d}}{\mathrm{d}y}\Big)\Big(\overline{\varepsilon}(s_1\Gamma_{3456}+s_2\Gamma_{3789})\Big)\Bigg)\Gamma\cdot X\Psi\\ &-i\int \mathrm{d}^4x\mathrm{Tr}\,\Bigg(\Big(\frac{q}{2e^2}+\frac{1}{e^2}\frac{\mathrm{d}}{\mathrm{d}y}\Big)\Big(\overline{\varepsilon}(t_1\Gamma_{3456}+t_2\Gamma_{3789})\Big)\Bigg)\Gamma\cdot Y\Psi.\end{aligned} \tag{2.46}$$

This can again be simplified using (2.32) and (2.33). The terms containing X become

$$\begin{aligned}\widetilde{\delta}I|_{2,X} &= -i\int \mathrm{d}^4x\mathrm{Tr}\,\Bigg(\Big(\frac{q}{2e^2}+\frac{1}{e^2}\frac{\mathrm{d}}{\mathrm{d}y}\Big)\big(\overline{\varepsilon}(-2\beta\Gamma_{3456}+2\gamma\Gamma_{3789}-q)\big)\Bigg)\Gamma\cdot X\Psi\\ &=-i\int \mathrm{d}^4x\frac{1}{e^2}\mathrm{Tr}\,\overline{\varepsilon}\Big((-2\beta'-\beta q+2\gamma\alpha)\Gamma_{3456}+(2\gamma'+\gamma q+2\beta\alpha)\Gamma_{3789}-q\alpha\Gamma_{0123}-\frac{q^2}{2}-q'\Big)\Gamma\cdot X\Psi\end{aligned} \tag{2.47}$$

The sum of $\widetilde{\delta}I'_X$ and $\widetilde{\delta}I|_{2,X}$ is

$$-i\,\mathrm{Tr}\int \mathrm{d}^4x\frac{1}{e^2}\overline{\varepsilon}\Big(\big(-2\beta'+4\gamma\alpha\big)\Gamma_{3456}+\big(2\gamma'+4\beta\alpha\big)\Gamma_{3789}+\big(2\beta^2+2\gamma^2-\frac{q^2}{2}-q'\big)\Big)\Gamma\cdot X\Psi. \tag{2.48}$$

The terms proportional to $\overline{\varepsilon}\Gamma_{0123}$ have canceled, but the terms involving $\overline{\varepsilon}\Gamma_{3456}$ and $\overline{\varepsilon}\Gamma_{3789}$ have not canceled. As a result, it is not in general possible to cancel (2.48) with an additional

contribution of the form (2.42). This is possible if and only if $\overline{\varepsilon}$ is an eigenvector of the matrix appearing in (2.48). We need

$$\overline{\varepsilon}\Big(\big(-2\beta' + 4\gamma\alpha\big)\Gamma_{3456} + \big(2\gamma' + 4\beta\alpha\big)\Gamma_{3789}\Big) = \overline{\varepsilon}\lambda, \tag{2.49}$$

where λ is a multiple of the identity. This condition is equivalent to the expected one (2.9), with ψ now determined in terms of α, β, γ.

Now let ε and $\widetilde{\varepsilon}$ be any two generators of the unbroken supersymmetry. Then $\overline{\varepsilon}\Gamma_3\widetilde{\varepsilon} = 0$, according to eqn. (2.10). So contracting (2.49) with $\Gamma_3\widetilde{\varepsilon}$ and expressing the result in terms of $B_1 = \Gamma_{3456}$, $B_2 = \Gamma_{3789}$, we get

$$0 = \overline{\varepsilon}\Gamma_3\Big(\big(-\beta' + 2\gamma\alpha\big)B_1 + \big(\gamma' + 2\beta\alpha\big)B_2\Big)\widetilde{\varepsilon}. \tag{2.50}$$

Let us decompose that in three terms

$$0 = \overline{\varepsilon}\Gamma_3\Big(-\beta' B_1 + \gamma' B_2\Big)\widetilde{\varepsilon} + \overline{\varepsilon}\Gamma_{0123}\Gamma_3\Big(\gamma B_2 - \beta B_1\Big)\alpha\widetilde{\varepsilon} + \overline{\varepsilon}\Gamma_3\Big(\gamma B_2 - \beta B_1\Big)\alpha\Gamma_{0123}\widetilde{\varepsilon}. \tag{2.51}$$

This recombines to

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(\overline{\varepsilon}\Gamma_3(\beta B_1 - \gamma B_2)\widetilde{\varepsilon}\right) = 0. \tag{2.52}$$

So we can integrate to give

$$\overline{\varepsilon}\Gamma_3(\beta B_1 - \gamma B_2)\widetilde{\varepsilon} = C, \tag{2.53}$$

with a constant C.

It is convenient to write the 16-dimensional space of positive chirality spinors (in which ε takes values) as $V_8 \otimes V_2$, where V_8 is an eight-dimensional space acted on by a double cover of $W = SO(1,2) \times SO(3)_X \times SO(3)_Y$, and V_2 is a two-dimensional space in which act the matrices B_0, B_1, and B_2 of eqn. (2.8). We take $\varepsilon = v \otimes \varepsilon_0$, $\widetilde{\varepsilon} = \widetilde{v} \otimes \widetilde{\varepsilon}_0$, with $v, \widetilde{v} \in V_8$, $\varepsilon_0, \widetilde{\varepsilon}_0 \in V_2$. The quadratic form $(\epsilon, \widetilde{\epsilon}) = \overline{\epsilon}\Gamma_3\widetilde{\epsilon}$ is symmetric in $\epsilon, \widetilde{\epsilon}$. It can be decomposed as the tensor product of an antisymmetric inner product in V_8 and an antisymmetric inner product in V_2. To write the inner product in V_2, we write ε_0 as a column vector

$$\varepsilon_0 = \begin{pmatrix} a \\ b \end{pmatrix}, \tag{2.54}$$

and define $\overline{\varepsilon}_0$ as a row vector:

$$\overline{\varepsilon}_0 = \begin{pmatrix} -b & a \end{pmatrix}. \tag{2.55}$$

Then the antisymmetric inner product in V_2 can be defined by

$$\langle \varepsilon_0, \widetilde{\varepsilon}_0 \rangle = \overline{\varepsilon}_0 \widetilde{\varepsilon}_0. \tag{2.56}$$

Equation (2.29) shows that the y evolution of ε_0 is just an $SO(2)$ rotation. Let us work in a basis in which B_0, B_1, and B_2 act as in eqn. (2.8), and normalize ε_0 at some value of y so that

$$\varepsilon_0 = \begin{pmatrix} \cos\,\psi/2 \\ \sin\psi/2 \end{pmatrix}, \tag{2.57}$$

for some ψ. Then (2.29) implies this form is valid for all y and moreover

$$\psi' = 2\alpha. \tag{2.58}$$

We set $\widetilde{\varepsilon}_0 = \varepsilon_0$, and plug the expression (2.57) into (2.53), with the result

$$\beta\cos\psi + \gamma\sin\psi = C \tag{2.59}$$

Now we can eliminate β and γ from equations (2.35), by contracting with $\Gamma_3\widetilde{\epsilon}$ for a conveniently chosen $\widetilde{\epsilon}$. If we take $\widetilde{\epsilon} = \widetilde{v}\otimes\widetilde{\varepsilon}_0$ with

$$\widetilde{\varepsilon}_0 = \begin{pmatrix} \cos\,3\psi/2 \\ -\sin\,3\psi/2 \end{pmatrix}, \tag{2.60}$$

then (2.35) reduces simply to

$$\begin{aligned} 0 &= 2\psi'\cos 2\psi + 2\beta\cos\psi + 2\gamma\sin\psi - q\sin 2\psi \\ &= 2\psi'\cos 2\psi - q\sin 2\psi + 2C. \end{aligned} \tag{2.61}$$

2.4.1 Exchange of $\vec{X}$ and $\vec{Y}$

We can repeat this analysis with $\vec{X}$ and $\vec{Y}$ exchanged, but it is more illuminating to observe that the problem has a symmetry that exchanges $\vec{X}$ and $\vec{Y}$. As a transformation of the underlying ten-dimensional spacetime, the relevant symmetry acts by $x^{3+i} \leftrightarrow x^{6+i}$, $i = 1, 2, 3$, together with a reflection of one of the coordinates x^0, x^1, x^2 (so as to preserve the overall orientation). On the above variables, the transformation exchanges β with γ and B_1 with B_2. It also changes the sign of α and maps ψ to $\pi/2 - \psi$. (This is implied by the relation $\alpha = d\psi/dy$ and the fact that the symmetry exchanges eigenvectors of B_1 with eigenvectors of B_2.) The formulas we obtain are symmetric in $\vec{X}$ and $\vec{Y}$, even though this is not manifest in the derivation. For example, the symmetry is present in (2.61).

2.5 Interpreting And Solving The Equations

According to (2.17), the supersymmetric Lagrangian has a three-dimensional Chern-Simons interaction, integrated in four dimensions:

$$\int \mathrm{d}^4x \frac{1}{e^2}\, u\epsilon^{\mu\nu\lambda}\mathrm{Tr}\,\left(A_\mu\partial_\nu A_\lambda + \frac{2}{3}A_\mu A_\nu A_\lambda\right). \tag{2.62}$$

Let us compare this to the θ-term of four-dimensional super Yang-Mills theory. This usually takes the form

$$I_\theta = -\frac{1}{32\pi^2}\int \mathrm{d}^4x\theta\epsilon^{\mu\nu\alpha\beta}\,\mathrm{Tr}\,F_{\mu\nu}F_{\alpha\beta}. \tag{2.63}$$

Usually, one assumes θ to be a constant and then the integral is a topological invariant. However, we wish to assume that θ is a function of $y = x^3$. Then, after integration by parts, we can write

$$I_\theta = \frac{1}{8\pi^2}\int \mathrm{d}^3xdy\frac{\mathrm{d}\theta}{\mathrm{d}y}\epsilon^{\mu\nu\lambda}\mathrm{Tr}\left(A_\mu\partial_\nu A_\lambda + \frac{2}{3}A_\mu A_\nu A_\lambda\right). \tag{2.64}$$

We see that we can interpret the combination u/e^2 as $\theta'/8\pi^2$.

On the other hand, in eqn. (2.40), we concluded that $u = -4\alpha$. So we have a more direct interpretation of α:

$$\alpha = -\frac{e^2\theta'}{32\pi^2}. \tag{2.65}$$

The other key equation governing the y-dependence of the couplings is (2.61):

$$-2\psi'\cos 2\psi + q\sin 2\psi = C. \tag{2.66}$$

Let us look for a domain wall solution in which the y coordinate extends over the whole real line and the coupling parameters e^2 and θ are both constant for $y \to \pm\infty$. Then ψ' and $q = -d\ln e^2/dy$ must vanish for $y \to \pm\infty$. This being so, the integration constant C must also vanish. So the equation reduces to

$$0 = -2\psi'\cos 2\psi + q\sin 2\psi. \tag{2.67}$$

Recalling that $\alpha = \psi'/2$, $q = e^2d(1/e^2)/dy$, we can rewrite eqns. (2.65) and (2.67) in the form:

$$\begin{aligned}\frac{\mathrm{d}\psi}{\mathrm{d}y} + \frac{e^2}{16\pi^2}\frac{\mathrm{d}\theta}{\mathrm{d}y} &= 0\\ -2\frac{\mathrm{d}\psi}{\mathrm{d}y}\cos 2\psi + e^2\sin 2\psi\frac{\mathrm{d}}{\mathrm{d}y}\frac{1}{e^2} &= 0.\end{aligned} \tag{2.68}$$

These equations have the remarkable property of being invariant under reparametrization of y.

Perhaps more to the point, we can solve them. (2.67) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(-\ln\sin 2\psi + \ln(1/e^2)\right) = 0, \tag{2.69}$$

so it says that

$$\frac{1}{e^2} = D \sin 2\psi, \tag{2.70}$$

with some constant D. Then we have $\mathrm{d}\theta/\mathrm{d}y = -(16\pi^2/e^2)d\psi/dy = -16\pi^2 D \sin 2\psi d\psi/dy = d(8\pi^2 D \cos 2\psi)/dy$. So we get

$$\theta = 2\pi a + 8\pi^2 D \cos 2\psi, \tag{2.71}$$

with another integration constant a.

The results for θ and $1/e^2$ are conveniently expressed in terms of the usual τ parameter

$$\tau = \frac{\theta}{2\pi} + \frac{2\pi i}{e^2}, \tag{2.72}$$

which takes values in the upper half plane. We have

$$\tau = a + 4\pi D(\cos 2\psi + i \sin 2\psi). \tag{2.73}$$

Thus, τ takes values in a circle of radius $4\pi D$, centered at the point $\tau = a$ on the real τ axis. (Just half of this circle is in the upper half plane.) Curves of this type are precisely the geodesics on the upper half-plane, with its standard $SL(2,\mathbb{R})$-invariant metric.[3] This unexpected appearance of $SL(2,\mathbb{R})$ symmetry means that our results are compatible with what is found in supergravity [1, 2, 4], where $SL(2,\mathbb{R})$ symmetry is manifest.

Now we can classify half-BPS domain walls of this type. We pick any two points τ_- and τ_+ in the upper half-plane and look for a domain wall with the property that $\tau(y) \to \tau_\pm$ for $y \to \pm\infty$. Any two points τ_+ and τ_- in the upper half plane are connected by a unique geodesic L, and the trajectory $\tau(y)$ must lie on L for all y. We gain absolutely no information about the function $\tau(y)$ except that its image lies on L and that the limits for $y \to \pm\infty$ are $\tau_\pm$. In particular, since the equations are invariant under reparametrization of y, there is no restriction on how the path from τ_- to τ_+ should be parametrized.

2.5.1 Solving For The Remaining Variables

From now on, we will keep $C = 0$. We return to eqn. (2.35), and contract with $\Gamma_3\varepsilon_0$. The result is

$$\psi' + \beta \cos\psi - \gamma \sin\psi = 0. \tag{2.74}$$

[3] A quick way to show this is to observe that the line $\mathrm{Re}\,\tau = 0$ is certainly a geodesic, since it is the fixed line of the isometry $\tau \to -\overline{\tau}$. Every geodesic is the image of this one under an $SL(2,\mathbb{R})$ transformation. On the other hand, an $SL(2,\mathbb{R})$ transformation maps the line $\mathrm{Re}\,\tau = 0$ to a semi-circle in the upper half-plane.

Combining this with the $C = \beta\cos\psi + \gamma\sin\psi = 0$ we finally get β, γ:

$$\begin{aligned} \beta &= -\frac{\psi'}{2\cos\psi} \\ \gamma &= \frac{\psi'}{2\sin\psi}. \end{aligned} \tag{2.75}$$

From the ansatz (2.57) for ε_0 and the explicit form of the matrices B_1 and B_2, we get

$$\bar{\varepsilon}\Gamma^3 \left(\sin\psi B_1 + \cos\psi B_2 + 1\right) = 0. \tag{2.76}$$

By acting on (2.32) with Γ^3 and comparing to the last equation, we learn that

$$s_1 + 2\beta = -q\sin\psi = -2\psi'\cos\psi + \frac{\psi'}{\cos\psi} \tag{2.77}$$

and

$$s_2 - 2\gamma = -q\cos\psi = +2\psi'\sin\psi - \frac{\psi'}{\sin\psi} \tag{2.78}$$

Hence

$$s_1 = 2\psi'\frac{\sin^2\psi}{\cos\psi}, \quad s_2 = 2\psi'\sin\psi. \tag{2.79}$$

Similarly,

$$t_1 = -2\psi'\cos\psi, \quad t_2 = -2\psi'\,\frac{\cos^2\psi}{\sin\psi}. \tag{2.80}$$

The eigenvalue λ in eqn. (2.49) turns out to be $\lambda = (d/dy)\left(\psi'/\sin\psi\cos\psi\right)$. Finally, we can solve for r

$$r = \lambda + 2\beta^2 + 2\gamma^2 - \frac{q^2}{2} - q' = 2\left(\psi'\tan\psi\right)' + 2(\psi')^2 \tag{2.81}$$

and by symmetry

$$\widetilde{r} = -2\left(\psi'\cot\psi\right)' + 2(\psi')^2 \tag{2.82}$$

2.5.2 Conformally Invariant Limit

Now (generalizing section 6 of [5]) we would like to ask whether, classically, it is possible to take a limit in which the Janus configuration becomes conformally invariant. As we have presented it so far, this configuration involves an arbitrary parametrization $\tau(y)$ of an arc in the upper half plane. To achieve conformal invariance (which acts by rescaling of y), $\tau(y)$ should simply have a discontinuity, say $\tau(y) = \tau_-$ for $y < 0$ and $\tau(y) = \tau_+$ for $y > 0$. When this is the case, $q = (d/dy)\ln(1/e^2)$ and θ' have delta function singularities; terms in

the action linear in q or θ' give contributions to the action supported at the interface. After integration by parts, the same is so for terms linear in q' or θ''. But contributions proportional to q^2 or $(\theta')^2$ are divergent in the conformally invariant limit. Our above formulas contain such terms, in view of the formulas for r and $\widetilde{r}$.

In the absence of a varying θ angle, this problem can be avoided [5] by a position-dependent rescaling of scalar fields. The same is possible in our case. After integration by parts of the ψ'' term, the rX^2 part of the action becomes

$$\frac{1}{2e^2}\mathrm{Tr}\,\left(-4\psi'\tan\psi\,X^aX'_a+2(\psi')^2\tan^2\psi\,X^aX_a\right). \tag{2.83}$$

This combines with the $(\partial_y X)^2$ term to a perfect square

$$\frac{1}{e^2}\mathrm{Tr}\,(X'-\psi'\tan\psi X)^2. \tag{2.84}$$

Similarly for the Y^2 terms, one gets

$$\frac{1}{e^2}\mathrm{Tr}(Y'+\psi'\cot\psi Y)^2. \tag{2.85}$$

If we define new scalar fields $\widetilde{X}=X\cos\psi$ and $\widetilde{Y}=Y\sin\psi$, the action simplifies. The terms just written become simply

$$\frac{1}{e^2}\mathrm{Tr}\left(\frac{(d\widetilde{X}/dy)^2}{\cos^2\psi}+\frac{(d\widetilde{Y}/dy)^2}{\sin^2\psi}\right). \tag{2.86}$$

Both $(\psi')^2$ and ψ'' disappear from the action, which becomes linear in ψ'. Hence the action has a well-defined limit to a localized discontinuity in τ.

Furthermore, $s_1/s_2=t_1/t_2=\tan\psi$. This means that the combination of gamma matrices which appears in the extra term (2.15) in the supersymmetry transformation is proportional to $B_1\sin\psi+B_2\cos\psi$, which leaves ε invariant. So the correction to the supersymmetry transformation is

$$\widetilde{\delta}\Psi=-\Gamma^3\Gamma\cdot X\psi'\tan\psi\,\varepsilon+\Gamma^3\Gamma\cdot Y\psi'\cot\psi\,\varepsilon. \tag{2.87}$$

We can combine this with the similar term in the unperturbed supersymmetry variation $\delta\Psi\sim\Gamma_3\Gamma\cdot(dX/dy)\varepsilon+\Gamma_3\Gamma\cdot(dY/dy)\varepsilon+\dots$ to

$$\delta'\Psi=\Gamma^3\left(\frac{\Gamma\cdot\widetilde{X}'}{\cos\psi}+\frac{\Gamma\cdot\widetilde{Y}'}{\sin\psi}\right)\varepsilon. \tag{2.88}$$

The structure that we have just found will be more apparent from a different viewpoint explained in section 3.4.

3 $3d$ Superfield Method

3.1 Overview

Our computation in section 2 was based on assuming the relevant R-symmetry and adjusting the couplings to achieve supersymmetry. Here, we will follow a different approach, using superfields to make manifest $\mathcal{N}=1$ supersymmetry and then adjusting the couplings to achieve R-symmetry – which then implies the full $\mathcal{N}=4$ supersymmetry.

But instead of merely repeating the problem studied in section 2 with a different approach, we will here study several closely related problems. So first we give an overview of the contents of this section.

3.1.1 A Three-Dimensional Problem

It is simplest to start with a purely three-dimensional problem. From a three-dimensional point of view, the generalized Janus configuration proposed in the previous section contains a Chern-Simons interaction as in eqn. (2.62). Of course, this configuration also has $\mathcal{N}=4$ supersymmetry in the three-dimensional sense (eight supercharges, not counting superconformal symmetries). This suggests that our subject is related to the problem of three-dimensional Chern-Simons interactions with $\mathcal{N}=4$ supersymmetry, and that will turn out to be the case.

As we recalled in the introduction, in three-dimensional nonabelian gauge theory with a Chern-Simons term, it is difficult to get past $\mathcal{N}=3$ supersymmetry if a conventional F^2 kinetic energy is present.[4] However, it has been argued [10] that one can achieve more supersymmetry in the absence of the F^2 term, and an example with $\mathcal{N}=8$ has been constructed [11].

[4]The standard argument for this (for example, see [6]) is based on the structure of the supermultiplets. In $2+1$ dimensions, the rotation group $SO(2)$ is abelian, and the spin of a particle is an integer or half-integer, either positive or negative. In the presence of both an F^2 term and a Chern-Simons interaction, the gauge fields become massive [13], say with spin 1. The $\mathcal{N}=3$ supersymmetry algebra has three spin lowering operators, and one can construct a supersymmetric theory of gauge fields, scalars, and fermions in which the gauge field is contained in a supermultiplet of states with spins $1, 1/2, 0, -1/2$. For $\mathcal{N}=4$, one would need also states of spin -1, which would also have to arise from gauge fields. Being in the same supermultiplet, the spin 1 and spin -1 gauge fields would have to transform the same way under the gauge group, but this is not possible for gauge fields in nonabelian gauge theory (gauge fields transform as precisely one copy of the adjoint representation). In $U(1)\times U(1)$ gauge theory, it is possible [8] to make an $\mathcal{N}=4$ theory with F^2 and Chern-Simons interactions; one $U(1)$ gauge boson has spin 1 and the other has spin -1. This is possible because, as $U(1)\times U(1)$ is abelian, the two gauge bosons both transform trivially under the gauge group.

In section 3.2, making no *a priori* assumptions about the appropriate gauge group or matter representations, we describe a general $\mathcal{N} = 4$ superconformal theory of this type. Our method is to assume $\mathcal{N} = 1$ superconformal symmetry, and adjust the couplings to find an $SO(4)$ R-symmetry that ensures that the model actually has $\mathcal{N} = 4$ superconformal invariance. We achieve a nice classification of models of this type. They correspond to supergroups in which the fermionic generators transform in a pseudoreal or symplectic representation of the bosonic symmetries. Apart from examples with abelian gauge group, the main examples involve the classical supergroups $U(N|M)$ (and its cousins $SU(N|M)$ and $PSU(N|N)$) and $OSp(N|M)$.

3.1.2 Intersecting Branes

These examples are related in an interesting way to a certain familiar configuration in string theory. Consider a system of parallel D3-branes ending on an NS5-brane from left and right – say N from the left and M from the right, as in fig. 1. We suppose that the D3-brane world-volumes are parametrized by x^0, x^1, x^2, x^3, while the NS5-brane world-volume is parametrized by x^0, x^1, x^2 and x^4, x^5, x^6. As usual, we set $y = x^3$. The physics of this configuration is well-known. For $y < 0$, we have $U(N)$ gauge theory; for $y > 0$, we have $U(M)$ gauge theory. At $y = 0$, there are bifundamental hypermultiplets, transforming in the representation $(N, \overline{M}) \oplus (\overline{N}, M)$ of $U(N) \times U(M)$.

This configuration is half-BPS, that is, it preserves eight supercharges (enhanced to 16 in the infrared limit, where the gauge theory becomes superconformal). It remains half-BPS if one turns on a ten-dimensional string theory axion field (the supersymmetric partner of the dilaton), inducing a four-dimensional theta-angle. However, it seems that in the literature, the low energy field theory representing the configuration of fig. 1 in the presence of a theta-angle is not known. As we will explain, constructing this field theory is very similar to constructing the supersymmetric Chern-Simons actions of section 3.2.

A theta-angle in four-dimensional gauge theory on a half-space is equivalent, classically, to a Chern-Simons interaction on the boundary, via a simple integration by parts:

$$-\frac{\theta}{32\pi^2}\int_{\mathcal{M}_+} \mathrm{d}^4x\, \epsilon^{\mu\nu\alpha\beta}\, \mathrm{Tr}\, F_{\mu\nu}F_{\alpha\beta} = \frac{\theta}{8\pi^2}\int_{\partial\mathcal{M}_+} \mathrm{d}^3x \epsilon^{\mu\nu\lambda} \mathrm{Tr}\, \left(A_\mu\partial_\nu A_\lambda + \frac{2}{3}A_\mu A_\nu A_\lambda\right). \quad (3.1)$$

($\mathcal{M}_+$ is the region $y > 0$, and $\partial\mathcal{M}_+$ is its boundary.) So the brane configuration of fig. 1 leads to $U(N) \times U(M)$ gauge theory coupled to three-dimensional bifundamental hypermultiplets, with three-dimensional Chern-Simons couplings for the $U(N)$ and $U(M)$ gauge fields. The Chern-Simons couplings have opposite signs, coming from integration by parts from left or right.

This is exactly the structure of one of the main examples of section 3.2, the one related

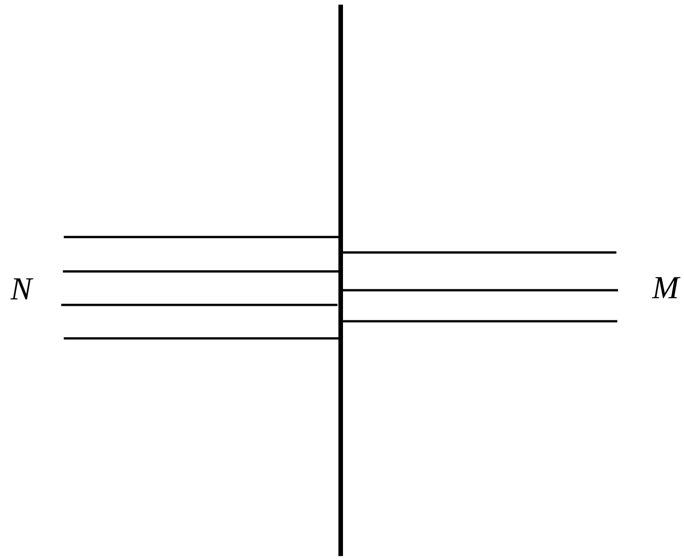

Figure 1: A configuration with N D3-branes ending on an NS5-brane from the left, while M D3-branes end from the right. The D3-brane worldvolumes span the 0123 directions, and those of the NS5-branes span the 012456 directions. The horizontal direction in the figure represents spacetime direction x^3, and the vertical direction represents spacetime directions 456.

to the supergroup $U(N|M)$. The only difference is that the $U(N)$ and $U(M)$ gauge fields live in four-dimensional half-spaces, while in section 3.2 they were purely three-dimensional.

The other main example from section 3.2 is the supergroup $OSp(N|M)$, which corresponds to gauge group $O(N) \times Sp(M)$, again with bifundamental matter. This example also arises naturally from a brane construction. In fact, one simply has to modify the brane construction of fig. 1 by including an O3-plane, parallel to the D3-branes. The gauge group of the D3-branes is then orthogonal or symplectic, and jumps from one type to the other in crossing the NS5-brane. This gives $O(N) \times Sp(M)$ gauge theory (with bifundamental hypermultiplets at the brane intersection), as in the $OSp(N|M)$ example from section 3.2.

The fact that the brane configurations give the same gauge groups and matter representations as the Chern-Simons theories is too much to be a coincidence, so it is reasonable to think that the relevant supersymmetric field theories can be constructed in the same way. Demonstrating this will be one of our goals.

3.1.3 Back To Janus

But what does all this have to do with the generalized Janus configurations of section 2?

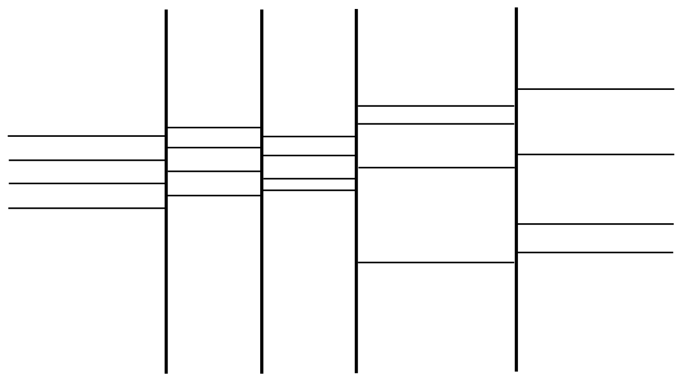

Figure 2: A system of N parallel D3-branes intersecting successive NS5-branes.

The brane configuration of fig. 1 has different "branches" of supersymmetric vacua. If one displaces the D3-branes incident from left and right in the x^4, x^5, x^6 directions (as is actually sketched in the figure), then the bifundamental hypermultiplet fields H become massive. However, it is also possible to give expectation values to the fields H. This corresponds to displacing some of the D3-branes normal to the NS5-brane by moving them in the directions x^7, x^8, x^9 – or by moving the NS5-brane in those directions. If $N = M$, the case we will now focus on, then it is possible to detach the NS5-brane from all D3-branes. The D3-branes that formerly ended from left or right on the NS5-brane instead reconnect to each other, and at low energies one is left with $\mathcal{N} = 4$ super Yang-Mills theory with gauge group $U(N)$.

The relation to Janus comes in because it is possible to modify this process slightly. Our field theory analysis of the configuration of fig. 1 will show that it is possible while preserving supersymmetry for the $U(M)$ theory on the right of the NS5-brane and the $U(N)$ theory on the left to have different four-dimensional gauge couplings. Moreover, it is possible to pick any embedding of three-dimensional $\mathcal{N} = 4$ supersymmetry in the four-dimensional supersymmetry algebra, that is, any value of the angle ψ in eqn. (2.9), and (roughly) any four-dimensional theta-angle θ, to the left of the NS5-brane. (The values of ψ and θ on the right are then uniquely determined.)

What happens, then, if we set $N = M$ and give an expectation value to the hypermultiplet fields? We reduce to a four-dimensional $U(N)$ gauge theory, but now with a coupling constant that "jumps" in crossing the hyperplane $y = 0$. There also are angles ψ or θ to the left of this hyperplane, which jump in crossing the hyperplane in a way that turns out to be consistent with eqn. (2.73). In short, we reduce to precisely the generalized Janus domain wall described in section 2.5.2.

We can go farther in this direction. We consider (fig. 2) a system of N parallel D3-branes that intersect k successive NS5-branes. From a field theory point of view, in each

slab between two NS5-branes (or each half-space to the left or right of all branes) there is a four-dimensional $U(N)$ gauge theory. At each interface between two slabs (or half-spaces), there are bifundamental hypermultiplets. A supersymmetric configuration can be constructed with any values of the angles ψ and θ to the left of all the NS5-branes, and any independently chosen four-dimensional gauge couplings in each of the various segments. (The values of ψ and θ in the other regions are then uniquely determined, via eqn. (2.73).) After giving generic expectation values to the hypermultiplet fields, we reduce to a four-dimensional theory of $U(N)$ gauge couplings in the presence of k successive generalized Janus domain walls.

A generalized Janus configuration with an arbitrary y-dependence of the gauge coupling, as described in [5] at $\theta = 0$ and in section 2.5 in general, can be obtained as a limit of this. We take a suitable limit in which k becomes large, the NS5-branes are closely spaced, and the individual jumps in the gauge coupling are small.

3.1.4 Janus and Fivebranes

It is useful to make explicit the condition for a Janus configuration to preserve the same supersymmetry as a defect or boundary which is the field theory limit of a (p, q) fivebrane. The original system of D3-branes preserves 16 out of 32 supercharges of the type II string theory. Writing ε_1 and ε_2 for the supersymmetries of left-moving and right-moving string modes, the supersymmetries left unbroken by the D3-branes are characterized by

$$\varepsilon_2 = \Gamma_{0123}\varepsilon_1. \tag{3.2}$$

A (p, q) (anti)fivebrane extended along the 012456 directions imposes a further constraint which depends on the appropriate central charge $p\tau + q$. In terms of $t = \arg(p\tau + q)$, the condition is

$$\varepsilon_1 = -\Gamma_{012456}\left(\sin t\, \varepsilon_1 + \cos t\, \varepsilon_2\right). \tag{3.3}$$

In the presence of both types of branes, we have

$$\varepsilon_1 = \left(-B_1 \cos t + B_2 \sin t\right)\varepsilon_1. \tag{3.4}$$

Comparing this to the constraint (2.9) imposed by the Janus configuration, we discover that they are compatible if $t = \frac{\pi}{2} + \psi$. On the other hand, the Janus configuration also prescribes that

$$\tau = a + 4\pi D e^{2i\psi} \tag{3.5}$$

with real constants a, D. This condition defines a semicircle in the upper half plane, which intersects the real axis at two points $a \pm 4\pi D$.

The condition $\arg(p\tau+q)=\frac{\pi}{2}+\psi$ is equivalent to

$$\frac{\sin(\pi/2+\psi)}{\cos(\pi/2+\psi)}=\frac{\mathrm{Im}\,(p\tau+q)}{\mathrm{Re}\,(p\tau+q)}, \tag{3.6}$$

or

$$-\frac{\cos\psi}{\sin\psi}=\frac{4\pi D\,\sin 2\psi}{a+q/p+4\pi D\cos 2\psi}. \tag{3.7}$$

This condition is actually independent of ψ, and equivalent to

$$a=-q/p-4\pi D. \tag{3.8}$$

So the rightmost intersection of the semicircle with the real axis must be at $\tau=-q/p$. Provided this condition is obeyed, supersymmetry is preserved when a (p,q)-fivebrane is added to a Janus configuration, regardless of the value of ψ at the location of the fivebrane.

We can repeat the same exercise for a (p',q') fivebrane extended along the 012789 directions. (The symmetry exchanging directions 456 with 789 is discussed in section 2.4.1.) The requirement is then $t=\psi$ and the leftmost intersection of the semicircle with the real axis must be at $-q'/p'$.

Hence a Janus configuration in which a and D are rational numbers can be combined with a (p,q) fivebrane in a supersymmetric fashion, for two different values of p and q. One compatible fivebrane runs in the 012456 directions, and has $a+4\pi D=-q/p$. The other one runs in the 012789 directions, with $a-4\pi D=-q'/p'$.

The specific case of D5-branes, which corresponds to $p=0$, requires a special treatment. In this case, t is 0 or π (depending on the sign q), and (3.4) becomes $\varepsilon_1=\pm\Gamma_{3456}\varepsilon_1$. This is the supersymmetry of the original half-BPS Janus configuration [5], in which the coupling constant varies but the Yang-Mills theta-angle does not. This is the case that the trajectory is a vertical line in the upper half plane, corresponding to a limit of the semicircle in which $a,D\to\infty$ with $a-4\pi D$ (or $a+4\pi D$) fixed.

3.1.5 Organization Of This Section

In section 3.2, we construct purely three-dimensional theories with Chern-Simons couplings and $\mathcal{N}=4$ supersymmetry. The technique that we will use is the one that we will follow throughout this section: we use three-dimensional $\mathcal{N}=1$ superspace to construct a Chern-Simons theory with $\mathcal{N}=1$ supersymmetry, and then constrain the couplings so that an extra global symmetry appears, promoting $\mathcal{N}=1$ to $\mathcal{N}=4$.

In section 3.3, as a prelude to some of the other questions described above, we reformulate four-dimensional $\mathcal{N}=4$ super Yang-Mills in terms of three-dimensional $\mathcal{N}=1$ superfields.

In section 3.4, we use this method to recover the generalized Janus configuration of section 2.2. This involves a computation that is arguably simpler than the one of section 2.2. In this formulation, it is evident that the Janus configuration has a conformally invariant limit, as found with greater effort in section 2.5.2. In section 3.5, we use three-dimensional $\mathcal{N}=1$ superfields to analyze the low energy field theories associated with the brane configurations of figs. 1 and 2, justifying some claims that were made above.

3.2 $\mathcal{N}=4$ Chern-Simons Theory

For any gauge group G, and any hypermultiplet representation of G, there is a unique classical theory with $\mathcal{N}=3$ Chern-Simons couplings. Morally, we want to show that if the matter content and gauge group are picked carefully, the resulting classical theory will actually possess $\mathcal{N}=4$ supersymmetry, or better $OSp(4|4)$ superconformal symmetry. It is known that the $\mathcal{N}=3$ theory is superconformal quantum mechanically as well, and we expect the same to be true of the $\mathcal{N}=4$ theory.

In practice, because $\mathcal{N}=3$ superfields are not convenient, we find it useful to start from an $\mathcal{N}=1$ Chern-Simons theory coupled to matter and look for an enhancement to $\mathcal{N}=4$. The hallmark of $\mathcal{N}=4$ is an $SO(4)$ R-symmetry group under which the four supercharges transform in the vector representation. The subgroup of $SO(4)$ that leaves fixed one of the supercharges is therefore $SO(3)$. So we start with an $\mathcal{N}=1$ theory with an $SO(3)$ global symmetry. We try to adjust the couplings so that the $SO(3)$ is enhanced to an $SO(4)$ that does *not* commute with $\mathcal{N}=1$ supersymmetry. Instead, the $SO(4)$ together with the $\mathcal{N}=1$ supersymmetry generate a full $\mathcal{N}=4$ structure. In fact, the theory is really conformal at the classical level, so the $\mathcal{N}=1$ theory we start with really has symmetry $OSp(1|4)$, and the enhancement is to $OSp(4|4)$.

The mechanism for symmetry enhancement will be the following. The matter fields will be $\mathcal{N}=1$ superfields $\mathcal{Q}^I_A = Q^I_A + i\theta^\alpha\lambda^I_{\alpha\,A} + \dots$, where θ^α are superspace coordinates, Q and λ are bosonic and fermionic fields, and the index $A=1,2$ transforms in the two-dimensional representation of a group that we will call $SU(2)_d$, the double cover of the global symmetry group $SO(3)$ mentioned in the last paragraph. The theory will have Yukawa couplings that are schematically of the form $Q^2\lambda^2$. Generically, these couplings are not invariant under separate $SU(2)$ symmetries acting on only Q or only λ. Our strategy is to adjust the superspace couplings so that such a $SU(2)\times SU(2)$ symmetry appears; in view of the relation of $SU(2)\times SU(2)$ to $SO(4)$, this will enhance $\mathcal{N}=1$ supersymmetry to $\mathcal{N}=4$. (This $SU(2)\times SU(2)$ is a cover of the group $SO(3)_X\times SO(3)_Y$ of section 2.)

As explained in the last paragraph, we will here make the assumption that one factor of $SU(2)\times SU(2)$ acts only on Q, and the other factor acts only on λ. This ansatz is too

restrictive to include the recently proposed Lagrangian [11] for an $\mathcal{N}=8$ Chern-Simons theory with gauge group $SO(4)$. It may also omit interesting constructions with $\mathcal{N}=4$ supersymmetry.

Notation

We generally follow the conventions of [14], chapter 2, for the $\mathcal{N}=1$ superspace in three dimensions. We denote spinor indices as Greek indices $\alpha, \beta, \ldots$. We raise and lower indices with a matrix

$$C_{\alpha\beta} = -C^{\alpha\beta} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{3.9}$$

as

$$\lambda_\alpha = \lambda^\beta C_{\beta\alpha} \qquad \lambda^\alpha = C^{\alpha\beta}\lambda_\beta \qquad \lambda^2 = \frac{1}{2}\lambda^\alpha\lambda_\alpha = i\lambda^+\lambda^- \tag{3.10}$$

Several useful relations are listed in appendix A. Superspace coordinates will be denoted as θ^α. A basic real superfield is expanded as

$$\mathcal{Q} = Q + \theta^\alpha\lambda_\alpha - \theta^2 F_Q \tag{3.11}$$

For such a superfield, the kinetic term is

$$-\frac{1}{2}\int \mathrm{d}^2\theta\,(\partial_\alpha\mathcal{Q})^2 = -\frac{1}{2}\partial_\mu Q\partial^\mu Q + \frac{1}{2}\lambda^\alpha(i\partial\!\!\!/)_\alpha^{\ \beta}\lambda_\beta + \frac{1}{2}F_Q^2. \tag{3.12}$$

A real superpotential may be added:

$$\int \mathrm{d}^2\theta\, W(\mathcal{Q}) = W''(Q)\lambda^2 + W'(Q)F_Q. \tag{3.13}$$

To describe n hypermultiplets in terms of $\mathcal{N}=1$ superfields, we introduce $4n$ superfields $\mathcal{Q}_A^I$, $I=1,\ldots,2n$, $A=1,2$. The group $SU(2)_d$ acts on A, while $Sp(2n)$ acts on I. The gauge group G will act via a homomorphism to $Sp(2n)$, so the hypermultiplets form a quaternionic representation of G. The metric on the hypermultiplet space will be $\epsilon^{AB}\omega_{IJ}$, where ϵ^{AB} and ω_{IJ} are respectively $SU(2)$-invariant and $Sp(2n)$-invariant antisymmetric tensors. The structure constants will have a quaternionic form $\tau_{IJ}^m = T_I^{mK}\omega_{KJ}$, symmetric in IJ. (Here m runs over a basis of the Lie algebra $\mathfrak{g}$ of G.) $\mathcal{Q}$ obeys the natural reality condition $\mathcal{Q}_I^{\dagger A} = \epsilon^{AB}\omega_{IJ}\mathcal{Q}_B^J$.

The gauge multiplet consists of a superconnection Γ_α and Γ_μ entering supercovariant derivatives $\mathcal{D}_\alpha = \partial_\alpha - i\Gamma_\alpha$ and $\mathcal{D}_\mu = \partial_\mu - i\Gamma_\mu$. There is a constraint

$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = 2i\mathcal{D}\!\!\!\!/\,_{\alpha\beta} \tag{3.14}$$

and a definition of field strength

$$[\mathcal{D}_\alpha, \not{\mathcal{D}}_{\beta\gamma}] = C_{\alpha(\beta}\mathcal{W}_{\gamma)}. \tag{3.15}$$

In the Wess-Zumino gauge, the only superpartner of the gauge field is

$$\mathcal{W}_\alpha|_{\theta=0} = \chi_\alpha. \tag{3.16}$$

The gauge-invariant extension of the matter kinetic energy is obtained by replacing ordinary superspace derivatives by covariant ones. This gives

$$-\frac{1}{2}\int \mathrm{d}^2\theta(\mathcal{D}_\alpha \mathcal{Q}_A^I)^2 = \frac{1}{2}\epsilon^{AB}\left(-\omega_{IJ}\mathcal{D}_\mu Q_A^I \mathcal{D}^\mu Q_B^J + \omega_{IJ}\lambda_A^{I\alpha}(i\not{\mathcal{D}})_\alpha^\beta \lambda_{B\beta}^J + \omega_{IJ}F_{QA}^I F_{QB}^J + 2\lambda_{A\alpha}^I \tau_{IJ}^m \chi_m^\alpha Q_B^J\right). \tag{3.17}$$

The standard kinetic term for the gauge fields is

$$\int \mathrm{d}^2\theta\, \mathcal{W}^2 = \frac{1}{2}\chi^\alpha(i\not{\mathcal{D}})_\alpha^\beta\chi_\beta + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{3.18}$$

But for the moment we are interested in gauge theories in which this term is absent, and the gauge fields have only a Chern-Simons action. The Chern-Simons term is essentially $\frac{1}{2}\Gamma^\alpha\mathcal{W}_\alpha + \cdots$, where we omit some extra terms cubic and quartic in Γ^α. In the Wess-Zumino gauge, this reduces to

$$\frac{k^{mn}}{4\pi}\left(A_m \wedge dA_n + \frac{2}{3}A_m \wedge [A, A]_n - \chi_m^\alpha \chi_{\alpha n}\right), \tag{3.19}$$

where k^{mn} is an invariant quadratic form on the Lie algebra $\mathfrak{g}$ of G. This quadratic form must obey a suitable integrality condition in order for the quantum theory to be well-defined. (For example, if G is a product of simple and simply-connected factors, then k is an integer multiple of the Killing form for each factor.)

In the absence of the conventional kinetic term (3.18), the part of the action quadratic in χ is purely the mass term present in (3.19), so χ is an auxiliary field. χ also enters the Yukawa coupling $\chi Q\lambda$ in (3.17), so integrating out χ gives a $Q^2\lambda^2$ coupling. This coupling is not invariant under $SU(2) \times SU(2)$ (with the two factors acting separately on Q and λ). However, if the superpotential $W(\mathcal{Q})$ is homogeneous and quartic in $\mathcal{Q}$, then a $Q^2\lambda^2$ term, also not invariant under $SU(2) \times SU(2)$, arises when the auxiliary field F_Q is integrated out of (3.13). Our procedure will be to choose W so that the $Q^2\lambda^2$ terms add up to be $SU(2) \times SU(2)$ invariant. The rest of the action will be $SU(2) \times SU(2)$ invariant for any choice of W.

The most general possible form of W, granted that it should be homogeneous and quartic (for conformal invariance) and $G \times SU(2)_d$ invariant, is as follows:

$$W(\mathcal{Q}) = \frac{\pi}{3}t_{IJ;KS}\epsilon^{AB}\epsilon^{CD}\mathcal{Q}_A^I\mathcal{Q}_B^J\mathcal{Q}_C^K\mathcal{Q}_D^S. \tag{3.20}$$

The tensor $t_{IJ;KS}$ is antisymmetric in the first two indices and in the last two indices, and symmetric under exchange of the first two with the last two. The full $Q^2\lambda^2$ interaction is

$$\pi Q^I_A Q^J_B \lambda^{\alpha K}_{\dot{C}} \lambda^S_{\alpha\dot{D}} \left(\epsilon^{A\dot{C}}\epsilon^{B\dot{D}}\tau^m_{IK}\tau^n_{JS}k_{mn} + \frac{2}{3}t_{IJ;KS}\epsilon^{AB}\epsilon^{\dot{C}\dot{D}} + \frac{4}{3}t_{IK;JS}\epsilon^{A\dot{C}}\epsilon^{B\dot{D}}\right). \tag{3.21}$$

The first term comes from integrating out the auxiliary fermion χ, as described in the last paragraph. The rest comes from the superpotential, via (3.13). If in (3.21), we can antisymmetrize the expression $Q^I_A Q^J_B$ in A and B, then the result is proportional to ϵ_{AB} and has the full $SU(2) \times SU(2)$ symmetry. Hence the condition for $SU(2) \times SU(2)$ invariance is that the part proportional to $Q^{(I}_{(A} Q^{J)}_{B)}$ is zero:

$$\tau^m_{IK}\tau^n_{JS}k_{mn} + \tau^m_{JK}\tau^n_{IS}k_{mn} + \frac{4}{3}t_{IK;JS} + \frac{4}{3}t_{JK;IS} = 0. \tag{3.22}$$

By summing this equation over cyclic permutations of IJK, we can eliminate t and get a condition that involves τ only:

$$\tau^m_{(IJ}\tau^n_{K)S}k_{mn} = 0. \tag{3.23}$$

This fundamental identity is a strong requirement on the gauge group and matter representation. Happily, it is possible to understand this condition in detail, since it is equivalent to the Jacobi identity for the following super Lie algebra:

$$\begin{aligned} [M^m, M^n] &= f^{mn}_s M^s \\ [M^m, \lambda_I] &= \tau^m_{IJ}\omega^{JK}\lambda_K \\ \{\lambda_I, \lambda_J\} &= \tau^m_{IJ}k_{mn}M^n. \end{aligned} \tag{3.24}$$

Of these conditions, the first just says that $\mathfrak{g}$ is a Lie algebra, and the second is equivalent to the statement that the hypermultiplets furnish a representation of this Lie algebra. The interesting statement is the last one, which asserts that the Lie algebra $\mathfrak{g}$ can be extended to a super Lie algebra $\widehat{\mathfrak{g}}$ by adjoining fermionic generators associated with the hypermultiplet representation. In verifying that eqn. (3.24) does define a super Lie algebra, the only nontrivial condition to verify is the $\lambda\lambda\lambda$ Jacobi identity:

$$[\lambda_I, \{\lambda_J, \lambda_K\}] + [\lambda_J, \{\lambda_K, \lambda_I\}] + [\lambda_K, \{\lambda_I, \lambda_J\}] = 0. \tag{3.25}$$

A short calculation shows that this precisely coincides with (3.23).

Moreover, we can now solve for t; each solution of the fundamental identity (3.23) determines a solution to (3.22) as well:

$$t_{IJ;KS} = \frac{1}{4}\tau^m_{IK}\tau^n_{JS}k_{mn} - \frac{1}{4}\tau^m_{IS}\tau^n_{JK}k_{mn} \tag{3.26}$$

The quadratic form k^{mn} that controls the Chern-Simons couplings in eqn. (3.19) also appears in (3.23) and (3.24). This means that the object $\widehat{k} = (k^{mn}, \omega^{IJ})$ is an invariant (and nondegenerate) quadratic form on the super Lie algebra $\widehat{\mathfrak{g}}$. Hence we get a consistent $N = 4$ Chern-Simons theory for each choice of a supergroup whose fermionic generators form a (possibly reducible) quaternionic representation of the bosonic subgroup, together with an invariant nondegenerate quadratic form $\widehat{k}$. The Chern-Simons couplings are determined by the restriction of $\widehat{k}$ to the bosonic Lie algebra $\mathfrak{g}$.

For our purposes, the prime examples are the classical supergroups $U(N|M)$ (or their cousins $SU(N|M)$ and $PSU(N|N)$) and $OSp(N|M)$. In each case, the gauge group is a product $U(N) \times U(M)$ or $O(N) \times Sp(M)$, with equal and opposite Chern-Simons couplings in the two factors. The fermion fields are in bifundamental hypermultiplets. The connection to a certain brane construction was described in section 3.1.2.

It will be useful to specialize the Lagrangian to the $U(N|M)$ case. We write A_1 and A_2 for the gauge fields of $U(N)$ and $U(M)$. The hypermultiplets consist of a pair of $N \times M$ matrices $\mathcal{Q}_A$, $A = 1, 2$, whose bosonic and fermionic components we denote $Q_A, \psi_{\dot{A}}$. $\mathcal{Q}^\dagger_A$ will be the hermitian adjoint of $\mathcal{Q}^A$, and similarly for Q^A. The structure constants are most easily described by giving the moment maps

$$\mu^m_{AB} = Q^I_A Q^J_B \tau^m_{IJ} \tag{3.27}$$

for the actions of $U(N)$ and $U(M)$:

$$\mu^{(1)}_{AB} = Q^\dagger{}_{(A} Q_{B)} \qquad\qquad \mu^{(2)}_{AB} = Q_{(A} Q^\dagger{}_{B)}. \tag{3.28}$$

The fundamental identity (3.23) is also easier to understand when recast[5] as

$$k_{mn} \mu^m_{(AB} \mu^n_{CD)} = 0. \tag{3.29}$$

Indeed, due to the opposite signs of the Chern-Simons coefficients of $U(N)$ and $U(M)$, the identity follows from the obvious relation

$$\mathrm{Tr}\, Q^\dagger{}_{(A} Q_B Q^\dagger{}_C Q_{D)} - \mathrm{Tr}\, Q_{(A} Q^\dagger{}_B Q_C Q^\dagger{}_{D)} = 0. \tag{3.30}$$

The superpotential can be expressed in terms of the superfield $\mathcal{M}^m_{AB} = \mathcal{Q}^I_A \mathcal{Q}^J_B \tau^m_{IJ}$ whose leading component is the moment map:

$$W = \frac{\pi}{6} \epsilon^{AB} \epsilon^{CD} \mathcal{Q}^I_A \mathcal{Q}^J_B \mathcal{Q}^K_C \mathcal{Q}^S_D \tau^m_{IK} \tau^n_{JS} k_{mn} = \frac{\pi}{6} \epsilon^{AB} \epsilon^{CD} \mathcal{M}^m_{AC} \mathcal{M}^n_{BD} k_{mn}. \tag{3.31}$$

[5] We can equivalently write $k_{mn}\mu^m_{(AB}\mu^n_{C)D} = 0$; once this tensor is symmetrized in three of the indices $ABCD$, it becomes automatically symmetric in all four.

Informally, the superpotential is the square of the moment map. In the specific case of the $U(N|M)$ theory, this reads

$$\mathcal{W}_4 = \frac{\pi k}{6} \mathrm{Tr}\, \mathcal{Q}^A \mathcal{Q}^\dagger{}_A \mathcal{Q}^B \mathcal{Q}^\dagger{}_B - \frac{\pi k}{6} \mathrm{Tr}\, \mathcal{Q}^A \mathcal{Q}^\dagger{}_B \mathcal{Q}^B \mathcal{Q}^\dagger{}_A. \tag{3.32}$$

The scalar potential is proportional to

$$\mathrm{Tr}\, Q^{[A} Q^\dagger{}_A Q^{B]} Q^\dagger{}_{[B} Q^C Q^\dagger{}_{C]}. \tag{3.33}$$

The equation for a critical point of the superpotential says that

$$Q^A Q^\dagger_C Q^B = Q^B Q^\dagger_C Q^A \tag{3.34}$$

for all $A, B, C = 1, 2$, along with the hermitian adjoint statement

$$Q^\dagger_A Q^C Q^\dagger_B = Q^\dagger_B Q^C Q^\dagger_A. \tag{3.35}$$

In particular, the bilinear matrices $Q_A Q^\dagger_B$, which include the moment maps, commute with each other. Their eigenvalues can be interpreted as points in $\mathbb{R}^4$. We will see in a later section that this is an important consistency requirement for the brane picture.

There is one last calculation which we will find useful. Consider the scalar potential in general

$$\frac{2\pi^2}{9} \mu^m_{AB} \mu^{Bn}_C k_{ms} k_{np} Q^{AI} \tau^s_{IJ} \omega^{JK} \tau^p_{KT} Q^{CT}. \tag{3.36}$$

The part symmetric in AC involves a commutator of the gauge group structure constants. Let us apply some transformations to half of this term. First, separate the symmetric and antisymmetric parts: the symmetric part is

$$\frac{\pi^2}{9} \mu^m_{AB} \mu^{Bn}_C k_{ms} k_{np} f^{sp}_q \mu^{qAC}. \tag{3.37}$$

The antisymmetric part is

$$\frac{\pi^2}{9} \mu^m_{AB} \mu^{ABn} k_{ms} k_{np} Q^I_C \tau^s_{IJ} \omega^{JK} \tau^p_{KT} Q^{CT}. \tag{3.38}$$

We can use the fundamental identity to rearrange this to

$$\frac{2\pi^2}{9} \mu^m_{AB} \mu^{Bn}_C k_{ms} k_{np} Q^{IC} \tau^s_{IJ} \omega^{JK} \tau^p_{KT} Q^{AT}. \tag{3.39}$$

This combines symmetrically in AC with the remaining part of the original potential to give again the commutator. The final result is

$$\frac{\pi^2}{6} \mu^m_{AB} \mu^{Bn}_C k_{ms} k_{np} f^{sp}_q \mu^{qAC}. \tag{3.40}$$

It is clear that the potential will be zero if the moment maps commute as elements of the Lie algebra.

3.2.1 The Current Multiplet

The conserved current that generates the gauge symmetry is part of a short multiplet of $\mathcal{N}=4$ supersymmetry or of the superconformal symmetry $OSp(4|4)$. The lowest dimension operators in this multiplet are the moment maps μ_{AB} (we suppress the $\mathfrak{g}$ index), which transform as $\mathbf{3}\otimes\mathbf{1}$ under $SU(2)\times SU(2)$. In the conformal case, these fields satisfy a BPS bound: their dimension is 1 and equals the spin under the R-symmetry group. The first descendants, which we will call $j_{A\dot{B}}$, are fermionic fields of dimension 3/2 transforming as $\mathbf{2}\otimes\mathbf{2}$. For the case that we have been treating so far that the matter system consists of free hypermultiplets, we have

$$j^m_{A\dot{B}} = Q^I_A \lambda^J_{\dot{B}} \tau^m_{IJ}. \tag{3.41}$$

After some algebra, the "Yukawa coupling" can be expressed in terms of this operator in a manifestly $SU(2)\times SU(2)$-invariant form

$$\pi Q^I_A Q^J_B \lambda^{\alpha K}_{\dot{C}} \lambda^S_{\alpha\dot{D}} \epsilon^{AB}\epsilon^{\dot{C}\dot{D}} \tau^m_{IS}\tau^n_{JK} k_{mn} = \pi\epsilon^{AB}\epsilon^{\dot{C}\dot{D}} j^m_{A\dot{C}} j^n_{B\dot{D}} k_{mn}. \tag{3.42}$$

Let us reconsider in terms of these variables what happens when the free hypermultiplets $\mathcal{Q}$ are coupled to gauge fields represented by $\mathcal{N}=1$ supermultiplets. (This is useful as preparation for considering more general matter systems in section 3.2.2.) The coupling of the auxiliary fermion χ in the component Lagrangian is $\chi^\alpha j_{A\dot{B}\alpha}\epsilon^{A\dot{B}}$, and integrating it away produces a jj coupling that lacks R-symmetry:

$$\pi k_{mn} j_{A\dot{B}}\epsilon^{A\dot{B}} j_{C\dot{D}}\epsilon^{C\dot{D}}. \tag{3.43}$$

The superpotential

$$\frac{\pi}{6}\epsilon^{AB}\epsilon^{CD}\mathcal{M}^m_{AC}\mathcal{M}^n_{BD}k_{mn} \tag{3.44}$$

gives two contributions to the Yukawa couplings; the two superderivatives can act on different factors of $\mathcal{M}$ or on the same factor:

$$\frac{\pi}{6}\epsilon^{AB}\epsilon^{CD} j^m_{(A\dot{C})} j^n_{(B\dot{D})} k_{mn} + \frac{\pi}{3}\epsilon^{AB}\epsilon^{CD}\mu^m_{AC} O^n_{\dot{B}\dot{D}} k_{mn}. \tag{3.45}$$

The operator $O_{\dot{A}\dot{B}}$ is a fermion bilinear of dimension two and spin zero that is an additional member of the $\mathcal{N}=4$ current multiplet for the free fields:

$$O_{\dot{A}\dot{B}} = \lambda^I_{\dot{A}}\lambda^J_{\dot{B}}\tau_{IJ}. \tag{3.46}$$

The fundamental identity (3.29) can be subjected to two superderivatives to give

$$\mu^{mAB} O^n_{\dot{C}\dot{D}} k_{mn} + j^{m(A}_{\dot{C}} j^{B)n}_{\dot{D}} k_{mn} = 0 \tag{3.47}$$

This identity allows one to rewrite the $QQ\lambda\lambda$ "Yukawa" interaction, which is the sum of (3.43) and (3.45), as a bilinear expression in j (we omit the factor $\frac{\pi}{3}k_{mn}$) .

$$3j_{A\dot{B}}\epsilon^{A\dot{B}} j_{C\dot{D}}\epsilon^{C\dot{D}} + \epsilon^{AB}\epsilon^{CD} j^m_{A\dot{C}} j^n_{B\dot{D}} + \epsilon^{AD}\epsilon^{CB} j^m_{A\dot{C}} j^n_{B\dot{D}} - \epsilon^{AB}\epsilon^{CD} j^m_{A\dot{B}} j^n_{C\dot{D}} - \epsilon^{AB}\epsilon^{CD} j^m_{A\dot{D}} j^n_{C\dot{B}}. \tag{3.48}$$

Rearranging the two indices of a current in the third and fifth terms by the usual $U_{AB} = U_{BA} + \epsilon_{AB}\epsilon^{CD}U_{CD}$ the non-R-symmetric terms drop and one is left with a simple Yukawa:

$$\pi k_{mn} j^m_{A\dot{B}} j^{nA\dot{B}} \tag{3.49}$$

3.2.2 Chern-Simons Coupling to General $\mathcal{N} = 4$ Matter System

So far, we have constructed $\mathcal{N} = 4$ Chern-Simons theories by coupling a gauge field to an $\mathcal{N} = 4$ matter system that consists simply of some free hypermultiplets. This can be generalized to replace the free hypermultiplets with a sigma model in which the target space is a hyper-Kahler manifold X. An important special case is that X is the Higgs (or Coulomb) branch of a superconformal field theory with $OSp(4|4)$ symmetry. However, for much of the discussion, this is not required.

If X is the Higgs branch of a CFT, then X is conical and in addition there is an $SU(2)$ symmetry acting on X and rotating the three complex structures. However, in the following, we do not need to assume the existence of such a symmetry. The only symmetries that we will assume that act on scalar fields will be the symmetries that make up the gauge group G. The scalars will be fields Q^{IA} that are associated with local coordinates on X, but in contrast to the discussion of the free hypermultiplets, we do not assume any symmetry acting on the A index. We will simply look for an $SU(2)$ symmetry that acts only on the fermion fields $\lambda_{I\dot{A}}$, transforming the $\dot{A}$ index. This is enough to promote $\mathcal{N} = 1$ supersymmetry to $\mathcal{N} = 4$, or in the conformal case to promote $OSp(1|4)$ to $OSp(4|4)$. (In the conformal case, the $SU(2)$ that rotates the complex structures is part of $OSp(4|4)$.) The hyper-Kahler structure of X can be described by the existence of antisymmetric inner products ω_{IJ} and ϵ_{AB}.

The moment maps $\mu^m{}_{AB}$ (m being a $\mathfrak{g}$ index) are defined as follows. Let V^m, $m = 1, \ldots, \dim \mathfrak{g}$ be the vector fields on X generating the action of $\mathfrak{g}$. And let Ω_{AB}, $A, B = 1, 2$ be the three symplectic forms[6] of the hyper-Kahler manifold X. Then the functions $\mu^m{}_{AB}$ are characterized as follows:

$$d\mu^m{}_{AB} = i_{V^m}(\Omega_{AB}). \tag{3.50}$$

(Here i_V is the operation of contraction with a vector field V.) This condition plus $\mathfrak{g}$-invariance determines μ^m_{AB} uniquely if G is semi-simple. In general, there are undetermined additive constants in μ, which correspond physically to the possibility of adding Fayet-Iliopoulos D-terms for $U(1)$ gauge fields. The fundamental identity makes sense for this class of models:

$$k_{mn}\mu^m_{(AB}\mu^n_{CD)} = 0. \tag{3.51}$$

Here k is some invariant and nondegenerate quadratic form on $\mathfrak{g}$.

[6]They can be defined by $\Omega_{AB} = dQ^{IC} \wedge dQ^{JD}\omega_{IJ}\epsilon_{AC}\epsilon_{BD}$.

Similarly, we can define $\mathcal{N}=4$ descendants of the fields μ^m_{AB}. The first descendant j, the fermionic current, is proportional to λ times a first derivative of μ. But according to (3.50), the derivatives of μ are essentially the vector fields V^m, so we can write j in terms of V^m:

$$j^{m\,A\dot{B}} = V^{mIA}\lambda^{\dot{B}J}\omega_{IJ}. \tag{3.52}$$

The next descendant of μ is constructed from the second derivative of μ or equivalently from the first derivative of V. In general, if V is a Killing vector field on a Riemannian manifold, one has $D_I V_J + D_J V_I = 0$. On a hyper-Kahler manifold, with V assumed to preserve all three complex structures, one has a stronger version of this statement:

$$D_{IA}V^m_{JB} = \tau^m_{IJ}\epsilon_{AB}, \tag{3.53}$$

where τ^m_{IJ} is symmetric in I and J. In the case of free hypermultiplets, the τ^m are simply constant matrices (which generate the action of G on the hypermultiplets), but in general, they are tensor fields on X.

As in our study of the linear hypermultiplets, we construct the action starting with $\mathcal{N}=1$ superfields and then looking for an additional symmetry. Let us denote with $\mathcal{M}^{AB}$ again the $\mathcal{N}=1$ superfield whose lowest component is μ^{AB}. The components of $\mathcal{M}^{AB}$ are

$$\mathcal{D}_\alpha \mathcal{M}^{mAB}|_{\theta=0} = j^{AB}_\alpha \tag{3.54}$$

and

$$\mathcal{D}^\alpha \mathcal{D}_\alpha \mathcal{M}^{AB}|_{\theta=0} = \tau^{IJ}\lambda^{\alpha A}_I \lambda^B_{\alpha J} = O^{AB}. \tag{3.55}$$

(In these formulas, we suppress the $\mathfrak{g}$ index m.)

We can now repeat step by step the analysis done in the free field case, and every step is formally identical. The potentially non-R-invariant fermion bilinears are still given by (3.43) and (3.45), and they still add up to an R-invariant sum (3.42) if the moment map μ^m obeys the fundamental identity:

$$k_{mn}\mu^m_{(AB}\mu^n_{CD)} = 0. \tag{3.56}$$

In fact, we only need the weaker condition (3.47), which is a second descendant of the fundamental identity:

$$\mu^{mAB}O^n_{\dot{C}\dot{D}}k_{mn} + j^{m(A}_{\dot{C}}j^{B)n}_{\dot{D}}k_{mn} = 0 \tag{3.57}$$

Elsewhere, we will show that there actually are $\mathcal{N}=4$ models for which this weaker condition is obeyed (in fact, the first descendant of the fundamental identity vanishes) but the fundamental identity itself is not satisfied. However, in the superconformal case, there are superconformal lowering operators and (3.57) actually implies (3.56).

The superconformal transformations of the fundamental identity are of interest. The current multiplet is a short multiplet of $OSp(4|4)$, as the leading component μ^{AB} satisfies a BPS

bound: it has spin $(1,0)$ under the R-symmetry group and dimension 1. The fundamental identity has spin $(2,0)$ and dimension 2: it is the protected component of the product of two current multiplets. This fact will probably play a useful role in a quantum description of these theories.

3.2.3 Quivers

Thus, we have a general recipe to couple $\mathcal{N}=4$ Chern-Simons gauge fields to any hyper-Kahler manifold X that obeys the fundamental identity. Of course, this is only interesting if there are examples beyond the ones associated with free hypermultiplets. We will now describe a family of such examples. We begin with a special case.

We start with a symmetry group $U(N_1)\times U(N_2)\times U(N_3)$ acting on the following free hypermultiplets: we include hypermultiplets Y that transform as $(N_1,\overline{N}_2,1)$ plus complex conjugate, and hypermultiplets Z that transform as $(1,N_2,\overline{N}_3)$ plus complex conjugate. (We use the same symbols Y and Z to denote the hypermultiplets and the spaces that they parametrize.) We write μ_Y and μ'_Y for the hyper-Kahler moment maps for the action of $U(N_1)$ and $U(N_2)$ on Y, and likewise μ_Z and μ'_Z for the hyper-Kahler moment maps for the action of $U(N_2)$ and $U(N_3)$, respectively, on Z. The hyper-Kahler moment map for the action of $U(N_1)\times U(N_2)\times U(N_3)$ on $Y\times Z$ is therefore

$$\mu_{Y\times Z}=(\mu_Y,\mu'_Y+\mu_Z,\mu'_Z), \tag{3.58}$$

where the three components refer to the three factors. Now we let X denote the hyper-Kahler quotient $(Y\times Z)///U(N_2)$. We recall that the hyper-Kahler quotient is obtained by setting to zero the moment map for $U(N_2)$, that is by imposing

$$\mu'_Y+\mu_Z=0, \tag{3.59}$$

and dividing by $U(N_2)$. On the quotient, we still have an action of $U(N_1)\times U(N_3)$, and the moment map can be read off from (3.58):

$$\mu_X=(\mu_Y,\mu'_Z). \tag{3.60}$$

Now if W is any hyper-Kahler manifold with action of a group $U(N)$ (where N may be N_1, N_2, or N_3) with moment map μ, we set $f_{(ABCD)}(\mu)=\sum_m \mu^m{}_{(AB}\mu^m{}_{CD)}$. (The sum is taken in an orthonormal basis for the Killing form.) Since f, whose subscripts we will suppress, is homogeneous and quadratic, we have

$$f(\mu)=f(-\mu). \tag{3.61}$$

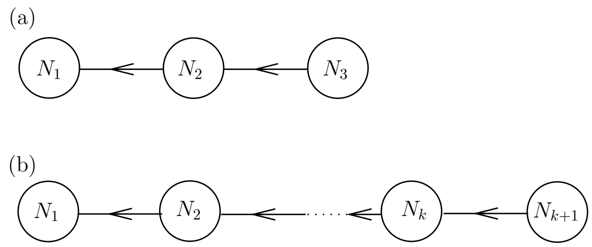

Figure 3: (a) A quiver associated with the first non-trivial example of a hyper-Kahler manifold obeying the fundamental identity. (b) More general linear quivers leading to solutions of the fundamental identity.

Since Y obeys the fundamental identity for the action of $U(N_1) \times U(N_2)$, we have

$$f(\mu_Y) - f(\mu'_Y) = 0. \tag{3.62}$$

(As usual, the minus sign reflects the structure of the invariant quadratic form $\widehat{k}$ of the supergroup $U(N_1|N_2)$.) Similarly, Z obeys the fundamental identity for the action of $U(N_2) \times U(N_3)$, so

$$f(\mu_Z) - f(\mu'_Z) = 0. \tag{3.63}$$

For the hyper-Kahler quotient X, we have the additional condition (3.59), which by virtue of (3.61) implies that $f(\mu'_Y) = f(\mu_Z)$. Combining these results, we learn that the hyper-Kahler moment map of X obeys

$$f(\mu_Y) - f(\mu'_Z) = 0. \tag{3.64}$$

This means that X obeys the fundamental identity for the action of $G = U(N_1) \times U(N_3)$, provided we take equal and opposite Chern-Simons levels for the two factors in G.

Thus, we get our first example of a non-flat hyper-Kahler manifold obeying the fundamental identity. The construction can be conveniently described via a simple quiver (fig. 3). A node in the quiver represents a unitary group $U(N)$ for some $N \geq 0$. A link connecting two nodes represents bifundamental hypermultiplets transforming under the given product of groups. The links are oriented, as indicated by the arrows. We pick a nonzero integer r and assign to each node a Chern-Simons level which is the product of r times the number of arrows entering the node minus the number of arrows leaving. Thus, in the example of fig. 3a, there are three nodes, with levels $r, 0, -r$. We take the hyper-Kahler quotient (of the space parametrized by the hypermultiplets) by the product of all groups associated with nodes of level 0. The result is a hyper-Kahler manifold X. It is acted on by a group G that is the product of the factors associated with nonzero levels.

The explicit example based on the product $U(N_1) \times U(N_2) \times U(N_3)$ is associated to the quiver of fig. 3a. The general case of a linear quiver, as in fig. 3b, is similar. The gauge

group G is a product of two unitary groups, associated with the ends of the quiver; these are the only nodes with nonzero labels. The fundamental identity is obeyed for the action of G on X, by virtue of essentially the same argument that we used for the quiver of fig. 3a. Moreover, essentially the same argument works for orthosymplectic quivers, in which the gauge groups are alternatively orthogonal or symplectic along the chain. Here one uses at each step the solution of the fundamental identity associated to $OSp(N|M)$.

Because of their origin as hyper-Kahler quotients of linear spaces, the spaces X obtained this way are actually conical, and have an $SU(2)$ action rotating the three complex structures. In fact, these examples are associated to superconformal field theories, which we will study in detail elsewhere.

3.3 $4d$ $\mathcal{N}=4$ Super Yang-Mills in a $3d$ Language

Given the simplicity of the Chern-Simons calculation, it is natural to wonder if a similar method can be applied to the Janus configuration, or to other modifications of $\mathcal{N}=4$ super Yang-Mills that preserve $\mathcal{N}=4$ supersymmetry in the three-dimensional sense.

As preparation, we will describe the undeformed $\mathcal{N}=4$ Lagrangian in the 3d $\mathcal{N}=1$ language. It will be then very simple to deform this to allow for various kinds of "defects," including y-dependent couplings (section 3.4), and interfaces between two possibly different gauge theories with bifundamental matter living at the interface (section 3.5). It is also possible to include defects that support extra hypermultiplets coupled to the bulk gauge fields [REFERENCE]. Any of these defects can coexist, provided that they preserve the same 8 supersymmetries.

From a three-dimensional point of view, the gauge field A_μ, $\mu=0,\ldots,3$, splits up as a three-dimensional gauge field A_μ, $\mu=0,1,2$, and an adjoint-valued scalar A_3. In terms of $3d$ $\mathcal{N}=1$ superfields, the three-dimensional gauge field is part of a standard superconnection Γ_α; this multiplet, which already appeared in section 3.2, also describes a fermion $\sigma_{A\alpha}$. (In the purely three-dimensional discussion, the analogous field was an auxiliary field and was called χ.) On the other hand, A_3 is the leading component of a real superfield

$$\mathcal{A}_3 = A_3 + \theta^\alpha\sigma_{3\alpha} + \theta^2 F_3^a \tag{3.65}$$

that transforms inhomogeneously under gauge transformations:

$$\mathcal{A}_3^g = g^{-1}\mathcal{A}_3 g - g^{-1}\partial_3 g. \tag{3.66}$$

The quantity that transforms most simply is the covariant derivative $\partial_3+\mathcal{A}_3$. The inhomogeneous term in (3.66) also changes the expression for the gauge-covariant superderivative

$$\mathcal{D}_\alpha\mathcal{A}_3 = D_\alpha\mathcal{A}_3 + \{\Gamma_\alpha,\mathcal{A}_3\} - \partial_3\Gamma_\alpha. \tag{3.67}$$

Accordingly, the component expansion of $\mathcal{D}_\alpha \mathcal{A}_3$ contains not the naive non-gauge invariant derivative $\partial_\mu A_3$ or covariant derivative $D_\mu A_3$, but the whole field strength $F_{\mu 3}$. Meanwhile, the scalar fields X^a are the leading components of real superfields

$$\mathcal{X}^a = X^a + \theta^\alpha \rho^a_{1\alpha} + \theta^2 F^a_X \tag{3.68}$$

which transform in the adjoint representation of the gauge group and as a **3** of the diagonal flavor symmetry group $SU(2)_d$. The same is true of Y^p:

$$\mathcal{Y}^p = Y^p + \theta^\alpha \rho^p_{2\alpha} + \theta^2 F^a_Y. \tag{3.69}$$

The fermions as described so far transform as $\mathbf{3} \oplus \mathbf{3} \oplus \mathbf{1} \oplus \mathbf{1}$ under $SU(2)_d$. To prove R-symmetry, we will have to reorganize the fermions as the sum of two copies of the representation $\mathbf{2} \otimes \mathbf{2}$ of the full R-symmetry group $SU(2) \times SU(2)$. Each **3** is related by $SU(2) \times SU(2)$ to a linear combination of the two **1**'s. Which linear combination appears will depend on how three-dimensional $\mathcal{N} = 4$ supersymmetry is embedded in four-dimensional $\mathcal{N} = 4$, that is, it will depend on the angle ψ in eqn. (2.9). Of course, as long as we consider the pure $\mathcal{N} = 4$ theory (as opposed to the generalizations that we introduce starting in section 3.4), supersymmetry will hold simultaneously for all values of ψ.

We will now write the four-dimensional $\mathcal{N} = 4$ theory in terms of these three-dimensional superfields. In this formalism, the part of the kinetic energy that involves derivatives in the 012 directions will come from what would usually be called kinetic energy terms in three dimensions. Terms involving derivatives in the x^3 direction will arise from two different sources, the peculiar gauge-covariant derivatives of $\mathcal{A}_3$ and some carefully chosen terms in the superpotential involving covariant derivatives in the x^3 direction.

Altogether, the kinetic terms in the action come from a superspace interaction

$$\frac{1}{e^2} \int \mathrm{d}^2\theta \, \mathrm{Tr} \left(-\mathcal{W}^2 + (D_\alpha \mathcal{A}_3)^2 + (D_\alpha \mathcal{X}^a)^2 + (D_\alpha \mathcal{Y}^a)^2 - 2\mathcal{Y}^a [\partial_3 + \mathcal{A}_3, \mathcal{X}^a] \right). \tag{3.70}$$

Expanding this in components, the 3d gauge action is

$$-\frac{1}{e^2} \int \mathrm{d}^2\theta \, \mathrm{Tr} \, \mathcal{W}^2 = \frac{1}{e^2} \mathrm{Tr} \left(\frac{1}{2} F_{\mu\nu} F^{\mu\nu} - i \sigma^\alpha_A \not{D}^\beta_\alpha \sigma_{A\beta} \right). \tag{3.71}$$

The kinetic energy for A_3 is

$$\frac{1}{e^2} \int \mathrm{d}^2\theta \, \mathrm{Tr} (D_\alpha \mathcal{A}_3)^2 = \frac{1}{e^2} \mathrm{Tr} \left(F_{\mu 3} F^{\mu 3} - i \sigma^\alpha_3 \not{D}^\beta_\alpha \sigma_{3\beta} - (F_A)^2 - 2\sigma^\alpha_3 D_3 \sigma_{A\alpha} \right). \tag{3.72}$$

The three-dimensional part of the kinetic energy for $\mathcal{X}^a$ reads

$$\frac{1}{e^2} \int \mathrm{d}^2\theta \, \mathrm{Tr} (D_\alpha \mathcal{X}^a)^2 = \frac{1}{e^2} \mathrm{Tr} \left((D_\mu X^a)(D^\mu X^a) - i \rho^{a\alpha}_1 \not{D}^\beta_\alpha \rho^a_{1\beta} - (F^a_X)^2 - 2\rho^{a\alpha}_1 [\sigma_{A\alpha}, X^a] \right) \tag{3.73}$$

and similarly for $\mathcal{Y}$

$$\frac{1}{e^2}\int \mathrm{d}^2\theta\,\mathrm{Tr}(D_\alpha\mathcal{Y}^a)^2=\frac{1}{e^2}\mathrm{Tr}\left((D_\mu Y^a)(D^\mu Y^a)-i\rho_2^{a\alpha}\not{D}^\beta_\alpha\rho^a_{2\beta}-(F^a_Y)^2-2\rho_2^{a\alpha}[\sigma_{A\alpha},Y^a]\right). \tag{3.74}$$

And the part of the kinetic energy involving x^3 derivatives of X and Y comes from

$$-\frac{2}{e^2}\int \mathrm{d}^2\theta\,\mathrm{Tr}\,\mathcal{Y}^a[\partial_3+\mathcal{A}_3,\mathcal{X}^a]=-\frac{2}{e^2}\mathrm{Tr}\,(D_3F^a_XY^a+X^a[F_A,Y^a]+D_3X^aF^a_Y+$$
$$+Y^a[\sigma_3^\alpha,\rho^a_{1\alpha}]+\rho_2^{a\alpha}[\sigma_{3\alpha},X^a]+\rho_2^{a\alpha}D_3\rho^a_{1\alpha}). \tag{3.75}$$

The sum of all of these terms gives the conventional four-dimensional kinetic energy for all fields.

In addition to eqn. (3.75), which may be regarded as a superpotential interaction from a three-dimensional point of view, we need a conventional cubic superpotential:

$$\mathcal{W}_3=\frac{\epsilon_{abc}}{e^2}\mathrm{Tr}\left(-\cos\psi\left(\frac{1}{3}\mathcal{X}^a[\mathcal{X}^b,\mathcal{X}^c]-\mathcal{X}^a[\mathcal{Y}^b,\mathcal{Y}^c]\right)+\sin\psi\left(\frac{1}{3}\mathcal{Y}^a[\mathcal{Y}^b,\mathcal{Y}^c]-\mathcal{Y}^a[\mathcal{X}^b,\mathcal{X}^c]\right)\right) \tag{3.76}$$

The angle ψ will ultimately coincide with the angle that appears in eqn. (2.9) characterizing the embedding of three-dimensional supersymmetry in four dimensions. For the moment, the main point is that we can get the same four-dimensional $\mathcal{N}=4$ theory for any choice of ψ. Finally, the four-dimensional theta-angle comes from

$$\frac{\theta}{4\pi^2}\int \mathrm{d}^2\theta\,\mathrm{Tr}\,D^\alpha\mathcal{A}_3W^\alpha=\frac{\theta}{4\pi^2}\mathrm{Tr}\left(F\wedge F+i\not{D}^{\alpha\beta}(\sigma_{A\alpha}\sigma_{3\beta})+\frac{1}{2}D_3(\sigma_A^\alpha\sigma_{A\alpha})\right) \tag{3.77}$$

(The terms other than $\mathrm{Tr}\,F\wedge F$ are total derivatives of gauge-invariant quantities.)

Let us now check that the sum of the above terms reproduces the standard $\mathcal{N}=4$ Lagrangian. We need to verify that by integrating away the auxiliary fields F, the quartic scalar potential and the x^3 part of the kinetic energy are reproduced, and that the correct Yukawa couplings arise as well. The superpotential in the three-dimensional sense is

$$\mathcal{W}=\int \mathrm{d}x^3\left(-\frac{2}{e^2}\mathrm{Tr}\,\mathcal{Y}^a[\partial_3+\mathcal{A}_3,\mathcal{X}^a]+\mathcal{W}_3\right). \tag{3.78}$$

Its derivatives evaluated at $\theta=0$ are

$$\begin{aligned}
-e^2\frac{\partial\mathcal{W}}{\partial\mathcal{X}^a}&=-2D_3Y^a+\cos\psi\epsilon_{abc}([X^a,X^b]-[Y^a,Y^b])+2\sin\psi\epsilon_{abc}[X^b,Y^c]\\
-e^2\frac{\partial\mathcal{W}}{\partial\mathcal{Y}^a}&=2D_3X^a-2\cos\psi\epsilon_{abc}[X^a,Y^b]+\sin\psi\epsilon_{abc}([X^b,X^c]-[Y^b,Y^c])\\
e^2\frac{\partial\mathcal{W}}{\partial\mathcal{A}_3}&=2[X^a,Y^a].
\end{aligned} \tag{3.79}$$

The quartic scalar potential and some parts of the kinetic energy arise by squaring these expressions, adding, and integrating over x^3. This process generates terms quartic in X and Y that are independent of ψ, and can easily be rearranged into the standard R-symmetric $\mathcal{N}=4$ quartic potential. There are dangerous non-R-symmetric cubic terms, but they are total derivatives

$$-\frac{1}{e^2}\cos\psi\, D_3\left(\epsilon_{abc}\mathrm{Tr}[X^a,X^b]Y^c\right)-\frac{1}{e^2}\sin\psi\, D_3\left(\epsilon_{abc}\mathrm{Tr}X^a[Y^b,Y^c]\right). \tag{3.80}$$

Finally, the quadratic terms give the contributions $(D_3X)^2$ and $(D_3Y)^2$ to the kinetic energy.

Next we can look at the Yukawa couplings. Dropping unnecessary indices for clarity, $-\frac{1}{e^2}X^a$ couples to

$$2[\rho_1^{a\alpha},\sigma_{A\alpha}]+2[\rho_2^{a\alpha},\sigma_{3\alpha}]+\cos\psi\epsilon_{abc}\left([\rho_1^{\alpha b},\rho_{1\alpha}^c]-[\rho_2^{\alpha b},\rho_{2\alpha}^c]\right)+2\sin\psi\epsilon_{abc}[\rho_1^{\alpha b},\rho_{2\alpha}^c], \tag{3.81}$$

while $-\frac{1}{e^2}Y^a$ couples to

$$2[\rho_2^{a\alpha},\sigma_{A\alpha}]-2[\rho_1^{a\alpha},\sigma_{3\alpha}]+\sin\psi\epsilon_{abc}\left([\rho_1^{\alpha b},\rho_{1\alpha}^c]-[\rho_2^{\alpha b},\rho_{2\alpha}^c]\right)-2\cos\psi\epsilon_{abc}[\rho_1^{\alpha b},\rho_{2\alpha}^c]. \tag{3.82}$$

We want to pair up ρ_1 and ρ_2 with linear combinations of σ_A and σ_3 in order to restore the full R-symmetry. The kinetic terms of the fermions are

$$-\frac{1}{e^2}\left(i\sigma_A^\alpha \not{D}_\alpha^\beta\sigma_{A\beta}+i\sigma_3^\alpha\not{D}_\alpha^\beta\sigma_{3\beta}+i\rho_1^{a\alpha}\not{D}_\alpha^\beta\rho_{1\beta}^a+i\rho_2^{a\alpha}\not{D}_\alpha^\beta\rho_{2\beta}^a\right) \tag{3.83}$$

and clearly constrain the possible linear combinations to be an $SO(2)$ rotation. The combinations will be

$$\sqrt{2}\Psi_1^{A\dot{B}}=\rho_1^{(AB)}+\epsilon^{AB}\left(\cos\psi\sigma_A-\sin\psi\sigma_3\right) \tag{3.84}$$
$$\sqrt{2}\Psi_2^{A\dot{B}}=\rho_2^{(AB)}+\epsilon^{AB}\left(\sin\psi\sigma_A+\cos\psi\sigma_3\right). \tag{3.85}$$

The Yukawa couplings of X and Y, namely

$$-\frac{1}{e^2}\mathrm{Tr}X^{AB}\left(\cos\psi\left([\Psi_1^{A\dot{C}},\Psi_{1\dot{C}}^B]-[\Psi_2^{A\dot{C}},\Psi_{2\dot{C}}^B]\right)+2\sin\psi[\Psi_1^{A\dot{C}},\Psi_{2\dot{C}}^B]\right) \tag{3.86}$$

and

$$-\frac{1}{e^2}\mathrm{Tr}Y^{\dot{C}\dot{D}}\left(\sin\psi\left([\Psi_1^{A\dot{C}},\Psi_{1\dot{C}}^B]-[\Psi_2^{A\dot{C}},\Psi_{2\dot{C}}^B]\right)-2\cos\psi[\Psi_1^{A\dot{C}},\Psi_{2\dot{C}}^B]\right), \tag{3.87}$$

are R-symmetric with this choice. If we further identify

$$\Psi=B_1\varepsilon_0\otimes\Psi_1+B_2\varepsilon_0\otimes\Psi_2 \tag{3.88}$$

we reproduce the standard kinetic terms and Yukawa couplings of $\mathcal{N}=4$ super Yang-Mills. (Appendix D contains some additional conventions and details.)

The remaining D_3 part of the fermion kinetic terms should pair up Ψ_1 and Ψ_2. Indeed there is a $-\frac{1}{e^2}\rho_2 D_3\rho_1$ term in eqn. (3.75), but the σ terms appear to be missing. To make things R-symmetric, we need an additional coupling $-(\sin\psi\sigma_A+\cos\psi\sigma_3)D_3(\cos\psi\sigma_A-\sin\psi\sigma_3)$, but what we see in (3.72) is $-\sigma_3 D_3\sigma_A$. Luckily, the last is equal to the former plus a total derivative $-\sin\psi\, D_3(\cos\psi\frac{1}{2}(\sigma_3\sigma_3-\sigma_A\sigma_A)+\sin\psi\sigma_3\sigma_A)$. The term proportional to the theta-angle is also a total derivative, clearly.

We have reproduced the standard $\mathcal{N}=4$ Lagrangian. As an extra check, notice that the only ψ dependence in the R-symmetric component action is in the total derivative terms, in agreement with the fact that the same $\mathcal{N}=4$ theory is invariant under each of the different $OSp(4|4)$ supergroups.

The various $\mathcal{N}=1$ fermionic superpartners are packaged together in R-symmetric combinations in a way which may appear quite obscure. It is useful to consider the opposite point of view: start from the $\mathcal{N}=4$ gauge multiplet and look at it as an $OSp(4|4)$ multiplet. Some tedious computations collected in the appendix D show that the fields are organized into two mirror $OSp(4|4)$ multiplets, which have the same general structure as a $3d$ current multiplet. One multiplet contains $Y^{AB}, \Psi_2, \cos\psi F_{3\mu}-\frac{1}{2}\sin\psi\epsilon_{\mu\nu\rho}F^{\nu\rho}, \partial_3 X^{\dot{A}\dot{B}}$ while the other contains $X^{\dot{A}\dot{B}}, \Psi_1, \sin\psi F_{3\mu}+\frac{1}{2}\cos\psi\epsilon_{\mu\nu\rho}F^{\nu\rho}, \partial_3 Y^{AB}$. Further reduction from $\mathcal{N}=4$ to $\mathcal{N}=1$ reproduces the detailed structure of the R-symmetric fermion combinations.

3.4 Generalized Janus, Again

Now it is straightforward to apply this method to build again the generalized Janus configuration of section 2.

A key step in verifying R-symmetry in the previous section was to integrate by parts to remove non-R-symmetric total derivatives. If we make the couplings[7] e^2 and θ_{YM} functions of x^3 and repeat the calculation, then $SU(2)\times SU(2)$ invariance is broken down to a diagonal subgroup $SU(2)_d$ by terms proportional to de^2/dx^3 and $d\theta_{YM}/dx^3$ that arise when we integrate by parts. Let us try to correct the Lagrangian to restore the symmetry. By dimensional reasoning and gauge invariance, the only possibility is to add a superpotential term that is bilinear in the scalar fields. Moreover, this term must be $SU(2)_d$-invariant. A bit of inspection shows that adding terms proportional to $\mathcal{X}^2$ or $\mathcal{Y}^2$ will do irreparable damage to R-symmetry. With some hindsight, and inspired by the results of section 2, we will add the following term to the Lagrangian:

$$\frac{2}{e^2}\int \mathrm{d}^2\theta\left(\psi'\frac{\sin\psi}{\cos\psi}\right)\mathrm{Tr}\mathcal{Y}^a\mathcal{X}^a=\frac{2}{e^2}\left(\psi'\frac{\sin\psi}{\cos\psi}\right)\mathrm{Tr}(F_X^aY^a+X^aF_Y^a+\rho_2^{a\alpha}\rho_{1\alpha}^a). \tag{3.89}$$

[7] We henceforth write θ_{YM} for the gauge theory theta-angle to avoid confusion with the odd superspace coordinates.

We will also assume the coupling dependence deduced in the earlier computation; we suppose that $\tau = \theta_{YM}/2\pi + 4\pi i/e^2$ takes the form

$$\tau = a + 4\pi D e^{2i\psi} \tag{3.90}$$

with real constants a and D. (Alternatively, instead of building in our prior knowledge of this, we could use $\mathcal{N} = 1$ superfields to give a new derivation of this result.)

The superpotential is now the sum of (3.78) and the correction term of eqn. (3.89):

$$\mathcal{W} = \int \mathrm{d}x^3 \left(-\frac{2}{e^2} \mathrm{Tr}\, \mathcal{Y}^a D_3 \mathcal{X}^a + \mathcal{W}_3 + \frac{2}{e^2} \left(\psi' \frac{\sin\psi}{\cos\psi} \right) \mathrm{Tr} \mathcal{Y}^a \mathcal{X}^a \right). \tag{3.91}$$

When we vary this with respect to X, we must integrate by parts the $Y D_3 X$ term. We encounter a derivative $\mathrm{d}(1/e^2)/\mathrm{d}x^3$, which we express in terms of $\psi' = d\psi/dx^3$ using (2.68). The gradient of the superpotential becomes

$$\begin{aligned}
-e^2 \frac{\partial \mathcal{W}}{\partial \mathcal{X}^a} &= -2D_3 Y^a - 2\psi' \frac{\cos\psi}{\sin\psi} Y^a + \cos\psi \epsilon_{abc}([X^a, X^b] - [Y^a, Y^b]) + 2\sin\psi \epsilon_{abc}[X^b, Y^c] \\
-e^2 \frac{\partial \mathcal{W}}{\partial \mathcal{Y}^a} &= 2D_3 X^a - 2\psi' \frac{\sin\psi}{\cos\psi} X^a - 2\cos\psi \epsilon_{abc}[X^a, Y^b] + \sin\psi \epsilon_{abc}([X^b, X^c] - [Y^b, Y^c]) \\
e^2 \frac{\partial W}{\partial \mathcal{A}_3^a} &= 2[X^a, Y^a]
\end{aligned} \tag{3.92}$$

Notice that the combinations $D_3 Y^a + \psi' \frac{\cos\psi}{\sin\psi} Y^a$ and $D_3 X^a - \psi' \frac{\sin\psi}{\cos\psi} X^a$ appear here, as in (2.84) and (2.85). It is useful to remember the results of section 2: in that formalism, it was possible to reabsorb both the X^2 and Y^2 terms in the component Lagrangian and the extra $\gamma \cdot X\epsilon$ terms in the supersymmetry transformations by rescaling the scalar fields as $\widetilde{X} = X \cos\psi$ and $\widetilde{Y} = Y \sin\psi$. Similarly here, the choice of the extra superpotential term is such that the terms in the gradient of the superpotential that are linear in X and Y can be expressed as $D_3 \widetilde{X}^a / \cos\psi$ and $-D_3 \widetilde{Y}^a / \sin\psi$.

The quartic potential for X and Y is unchanged. The dangerous non-R-symmetric cubic terms receive several contributions, but these add up to true total derivatives:

$$-\partial_3 \left(\frac{1}{e^2} \cos\psi \epsilon_{abc} \mathrm{Tr}[X^a, X^b] Y^c \right) \tag{3.93}$$

and

$$-\partial_3 \left(\frac{1}{e^2} \sin\psi\, \epsilon_{abc} \mathrm{Tr} X^a [Y^b, Y^c] \right). \tag{3.94}$$

The R-symmetric terms cubic in X are

$$-\frac{2}{3e^2} \frac{\psi'}{\cos\psi} \epsilon_{abc} \mathrm{Tr} X^a [X^b, X^c] + \frac{1}{3} \partial_3 \left(\frac{1}{e^2} \sin\psi \mathrm{Tr} X^a [X^b, X^c] \right). \tag{3.95}$$

The R-symmetric terms cubic in Y are

$$\frac{2}{3e^2}\frac{\psi'}{\sin\psi}\epsilon_{abc}\mathrm{Tr}Y^a[Y^b,Y^c]+\frac{1}{3}\partial_3\left(\frac{1}{e^2}\cos\psi\mathrm{Tr}Y^a[Y^b,Y^c]\right). \tag{3.96}$$

We see that the bosonic Lagrangian indeed agrees with the one in section 2 up to total derivatives.

The fermion bilinear terms are also R-symmetric up to a total derivative. The ρ bilinears are:

$$-\frac{2}{e^2}\rho_2^{a\alpha}D_3\rho_{1\alpha}^a+2\psi'\frac{1}{e^2}\frac{\sin\psi}{\cos\psi}\rho_2^{a\alpha}\rho_{1\alpha}^a. \tag{3.97}$$

The σ bilinears are

$$-\frac{2}{e^2}\sigma_3^\alpha D_3\sigma_{A\alpha}+\frac{2}{e^2}\psi'\sigma_A^\alpha\sigma_{A\alpha}. \tag{3.98}$$

The last term is the non-trivial term provided by the variable theta-angle $\theta_{YM}=2\pi a+8\pi^2 D\cos 2\psi$

$$\frac{\theta}{4\pi^2}\int \mathrm{d}^2\theta\,\mathrm{Tr}\,D^\alpha\mathcal{A}_3W^\alpha=\frac{\theta}{4\pi^2}\mathrm{Tr}\left(F\wedge F+i\not{D}^{\alpha\beta}(\sigma_{A\alpha}\sigma_{3\beta})+\frac{1}{2}D_3(\sigma_A^\alpha\sigma_{A\alpha})\right), \tag{3.99}$$

after integration by parts. From now on we will occasionally drop the spinor indices for clarity. To verify R-symmetry, it is useful to integrate the ρ bilinear by parts to make the derivative antisymmetric; this gives[8]

$$-\frac{1}{e^2}\rho_2^a\overleftrightarrow{D_3}\rho_1^a+2D\psi'\rho_2^a\rho_1^a. \tag{3.100}$$

We must verify that the σ bilinear can be written in the same form, with ρ_i replaced by the appropriate linear combinations of σ_3,σ_A. The derivative may act both on σ_3,σ_A and on the coefficients of the linear combination, and after some rearrangements the form we need to find is

$$-\frac{1}{e^2}\sigma_3\overleftrightarrow{D_3}\sigma_A+\frac{\psi'}{e^2}(\sigma_A\sigma_A+\sigma_3\sigma_3)+D\psi'\sin 2\psi(\sigma_A\sigma_A-\sigma_3\sigma_3)+2D\psi'\cos 2\psi\sigma_A\sigma_3. \tag{3.101}$$

Let us rearrange this a little more, by integrating the antisymmetric derivative of σ back to the form $\sigma_3D_3\sigma_A$ which appears in the Lagrangian (3.98). The result of the integration by parts cancels the term $2D\psi'\cos 2\psi\sigma_A\sigma_3$ and the rest combines to $\frac{2\psi'}{e^2}\sigma_A\sigma_A$, as in (3.98). Hence we have proved R-symmetry of the whole component Lagrangian.

Let us collect all the various non R-symmetric total derivatives that appeared in this calculation from the cubic bosonic terms, from the integration by parts of fermion bilinears,

[8] Here and in (3.101), we use $D=1/e^2\sin 2\psi$.

and from the theta-angle. They add up to

$$\frac{\mathrm{d}}{\mathrm{d}x^3}\left(-\frac{1}{e^2}\cos\psi\,\epsilon_{abc}\mathrm{Tr}\,[X^a,X^b]Y^c-\frac{1}{e^2}\sin\psi\,\epsilon_{abc}\mathrm{Tr}\,X^a[Y^b,Y^c]-\frac{1}{e^2}\mathrm{Tr}\,\sigma_A^\alpha\sigma_{3\alpha}\right.$$
$$\left.-\frac{1}{e^2}\mathrm{Tr}\,\rho_1^{a\alpha}\rho_{2\alpha}^a+\frac{\theta_{YM}}{8\pi^2}\mathrm{Tr}\,\sigma_A^\alpha\sigma_{A\alpha}\right). \tag{3.102}$$

This formula will be useful at the next step, when adding a boundary to the theory.

3.5 Bifundamental Defect

We are going to apply what we have learned to a problem described in section 3.1.2 and in fig. 1 – an NS5-brane with N D3-branes ending from the left and M from the right. This system has been much-studied at $\theta_{YM}=0$, and the resulting low energy physics is well-known. There is an $\mathcal{N}=4$ theory with gauge group $U(N)$ in the half-space $x^3\leq 0$, another $\mathcal{N}=4$ theory with gauge group $U(M)$ in the half-space $x^3\geq 0$, and there are bifundamental hypermultiplets supported on the hyperplane $x^3=0$ and interacting with the gauge fields on both sides. The problem also has a variant with an orientifold threeplane parallel to the D3-branes; the gauge group is then $SO(N)\times Sp(M)$, still with bifundamental hypermultiplets supported at $x^3=0$.

As far as we know, the low energy effective action describing this system at $\theta_{YM}\neq 0$ has not been elucidated in the literature. It is easy to explain why. The formula

$$-\frac{\theta_{YM}}{32\pi^2}\int_{M_+}\mathrm{d}^4x\,\epsilon^{\mu\nu\alpha\beta}\,\mathrm{Tr}\,F_{\mu\nu}F_{\alpha\beta}=\frac{\theta_{YM}}{8\pi^2}\int_{\partial M_+}\mathrm{d}^3x\epsilon^{\mu\nu\lambda}\mathrm{Tr}\left(A_\mu\partial_\nu A_\lambda+\frac{2}{3}A_\mu A_\nu A_\lambda\right) \tag{3.103}$$

(where M_+ is a half-space, and ∂M_+ is its boundary) shows that supersymmetrizing the interaction $\mathrm{Tr}\,F\wedge F$ in four-dimensional gauge theory on a half-space is very similar to supersymmetrizing the Chern-Simons interaction in three dimensions. How to do this in a theory with the equivalent of three-dimensional $\mathcal{N}=4$ supersymmetry has not been clear.

However, in section 3.2, we constructed $\mathcal{N}=4$ Chern-Simons couplings for precisely the relevant cases – $U(N)\times U(M)$ or $SO(N)\times Sp(M)$ gauge theory with bifundamental hypermultiplets. As we will see, the problem in which the two factors of the gauge group live on four-dimensional half-spaces $x^3\leq 0$ and $x^3\geq 0$ can be treated very similarly to the purely three-dimensional problem of section 3.2.

If θ_{YM} is of the form $2\pi q$, $q\in\mathbb{Z}$, then by an S-duality transformation, one can set θ_{YM} to zero, at the cost of replacing the NS5-brane with a $(1,q)$-fivebrane. So our analysis also governs a system of D3-branes ending from left and right on a $(1,q)$-fivebrane.

We will not assume *a priori* that the gauge groups and matter representations are the particular ones appropriate to the D3-NS5 system. But since we will find the same constraints as in section (3.2), this will turn out to be the case

3.5.1 Gauge Fields In A Half-Space

We closely follow the logic of section 3.4, constructing $\mathcal{N} = 4$ super Yang-Mills theory in terms of three-dimensional $\mathcal{N} = 1$ superfields. Now, however, our gauge fields are defined only in a half-space, say $x^3 \geq 0$. It is most simple to consider first a one-sided problem with gauge fields in only one half-space and with no hypermultiplets.[9] This corresponds, in terms of branes, to having D3-branes on only one side of an NS5-brane. Then in section 3.5.2, we generalize to include hypermultiplets. From the standpoint of branes, the generalization is relevant to the two-sided case with different gauge groups on the two sides, and hypermultiplets supported at $x^3 = 0$.

The main difference from the previous analysis is that the various non-R-symmetric total derivatives in the component Lagrangian cannot be discarded. They give boundary contributions at $x^3 = 0$. These boundary contributions will play a role similar to the terms that in section 3.2 were found by integrating out the auxiliary field χ; they combine with terms coming from the $\mathcal{N} = 1$ superpotential to give an R-symmetric action.

In the following analysis, one can permit e^2 and θ_{YM} to be x^3-dependent, as long as they are constrained by eqn. (3.90). The boundary contributions that we focus on here do not involve derivatives of e^2 and θ_{YM}, so it simply does not matter whether e^2 and θ_{YM} are constant. These boundary terms can be read off from (3.102), and are

$$-\frac{1}{e^2}\cos\psi\epsilon_{abc}\mathrm{Tr}[X^a, X^b]Y^c - \frac{1}{e^2}\sin\psi\epsilon_{abc}\mathrm{Tr}X^a[Y^b, Y^c] \tag{3.104}$$

and

$$-\frac{1}{e^2}\mathrm{Tr}\sigma_A^\alpha\sigma_{3\alpha} - \frac{1}{e^2}\mathrm{Tr}\rho_1^{a\alpha}\rho_{2\alpha}^a + \frac{\theta_{YM}}{8\pi^2}\mathrm{Tr}\sigma_A^\alpha\sigma_{A\alpha}. \tag{3.105}$$

As usual, we want to make the $\rho_1\rho_2$ term part of an R-symmetric interaction $\Psi_1\Psi_2$ by combining it with the appropriate bilinear in σ_3, σ_A. After doing this, the remaining truly non-R-symmetric terms are

$$-\frac{1}{e^2}\sin\psi\cos\psi\mathrm{Tr}(\sigma_3\sigma_3 - \sigma_A\sigma_A) - \frac{2}{e^2}\sin^2\psi\mathrm{Tr}\sigma_3\sigma_A + \frac{\theta_{YM}}{8\pi^2}\mathrm{Tr}\sigma_A\sigma_A. \tag{3.106}$$

[9] The case with constant e^2 and θ_{YM} and no hypermultiplets will be treated more directly elsewhere. The present approach has the advantages of letting e^2 and θ_{YM} vary, and of extending to the two-sided case.

In the presence of a boundary, the computation of the derivatives of the superpotential also needs to be re-examined. Formula (3.92) for $\partial\mathcal{W}/\partial\mathcal{X}^a$ receives an extra delta function contribution by integration by parts of the YD_3X contribution to $\mathcal{W}$. So one now has

$$-e^2\frac{\partial\mathcal{W}}{\partial\mathcal{X}^a}=-2D_3Y^a-2\psi'\frac{\cos\psi}{\sin\psi}Y^a+\cos\psi\epsilon_{abc}([X^a,X^b]-[Y^a,Y^b])+2\sin\psi\epsilon_{abc}[X^b,Y^c]+Y^a\delta(x^3). \tag{3.107}$$

The action will contain a term $\int \mathrm{d}x^3|\partial\mathcal{W}/\partial\mathcal{X}|^2$, and as we do not want a term proportional to $\int \mathrm{d}x^3\delta(x^3)^2$, we conclude that the boundary condition must be $\vec{Y}=0$.

This argument is a little disingenuous, since the underlying theory has a complete symmetry between $\vec{Y}$ and $\vec{X}$. Instead of including in $\mathcal{W}$ a term $-\int \mathrm{d}x^3\,\mathrm{Tr}\,\mathcal{Y}D_3\mathcal{X}$, we could have integrated by parts and included a term $\int \mathrm{d}x^3\,\mathrm{Tr}\,(D_3\mathcal{Y})\mathcal{X}$. This change would not have affected the reasoning in section 3.4, but an argument similar to the above[10] would now lead us to a boundary condition $\vec{X}=0$. Actually, the boundary condition $\vec{Y}=0$ is very natural for describing the D3-NS5 system of fig. 1. If the NS5-brane is characterized by $x^7=x^8=x^9=0$, then, as the scalar fields Y^p parametrize the position of the D3-branes in those directions, the boundary condition $\vec{Y}=0$ is natural. The boundary condition $\vec{X}=0$ is the one we want if the NS5-brane is characterized by $x^4=x^5=x^6=0$.

The boundary condition $\vec{Y}=0$ is extended by $\mathcal{N}=1$ supersymmetry to a superspace boundary condition $\vec{\mathcal{Y}}=0$. Using the superspace expansion of eqn. (3.69), this amounts to

$$0=Y^a=\rho_2^a=F_Y^a. \tag{3.108}$$

On the other hand, $F_Y=\partial\mathcal{W}/\partial Y$ has been computed in (3.92). Given the boundary condition $\vec{Y}=0$, the vanishing of F_Y at the boundary gives a boundary condition for $\vec{X}$:

$$0=D_3X^a-\psi'\frac{\sin\psi}{\cos\psi}X^a=\frac{D_3\widetilde{X}^a}{\cos\psi}, \tag{3.109}$$

with $\widetilde{X}^a=X^a\,\cos\psi$. This boundary condition must of course also be extended to a modified Neumann boundary condition on the superfield $\vec{\mathcal{X}}$.

Since we want $\mathcal{N}=4$ supersymmetry, not just $\mathcal{N}=1$ supersymmetry, we must extend (3.108) to a set of boundary conditions with the full $SU(2)\times SU(2)$ R-symmetry. In particular, the $SU(2)\times SU(2)$-symmetric extension of the boundary condition $\rho_2^a=0$ is to require also that

$$\sin\psi\sigma_A+\cos\psi\sigma_3=0 \tag{3.110}$$

[10] More generally, we could take $\mathcal{W}$ to be a more generic linear combination of the two expressions. Then to cancel delta function terms, we would be led to impose $\vec{X}=\vec{Y}=0$ on the boundary. This, however, is incompatible with preserving one-half of the supersymmetry. We will show this in more detail elsewhere, but a quick argument is as follows. Given that $\vec{Y}=0$ on the boundary, we will deduce below from supersymmetry that $\vec{X}$ must obey modified Neumann boundary conditions (3.109). It is therefore not possible for $\vec{X}$ to obey Dirichlet boundary conditions.

at the boundary. $\mathcal{N}=1$ supersymmetry then extends this to a further boundary condition

$$F_{3\mu}=\tan\psi\frac{1}{2}\epsilon_{\mu\nu\rho}F^{\nu\rho}. \tag{3.111}$$

Setting $Y=0$ automatically sets to zero the non-R-invariant bosonic terms in (3.104). What about the fermionic terms in (3.106)? Precisely if

$$\frac{\theta_{YM}}{2\pi}=-\frac{4\pi}{e^2}\frac{\sin\psi}{\cos\psi} \tag{3.112}$$

at $x^3=0$, (3.106) becomes a "perfect square"

$$\frac{1}{e^2}\tan\psi(\cos\psi\sigma_3+\sin\psi\sigma_A)^2, \tag{3.113}$$

and vanishes by virtue of the boundary condition (3.110). The condition (3.112) is necessary for this result, since it was needed to cancel a σ_A^2 coupling that is not R-symmetric.

Thus, we have established the full R-symmetry and hence $\mathcal{N}=4$ supersymmetry in the absence of hypermultiplets supported at $x^3=0$. Before going on to the more general case in section 3.5.2, we pause to interpret the relation (3.112) that was needed for this result.

Consider four-dimensional gauge fields with the action

$$I=\int \mathrm{d}^4x\left(\frac{1}{2e^2}F_{IJ}F^{IJ}-\frac{\theta_{YM}}{32\pi^2}\epsilon^{IJKL}\mathrm{Tr}F_{IJ}F_{KL}\right) \tag{3.114}$$

(here we take indices $I,J,\dots=0,\dots,3$). In this theory, "free" boundary conditions, in which the variation of the connection is unconstrained on the boundary, read

$$F_{3\mu}+\frac{e^2\theta_{YM}}{8\pi^2}\frac{1}{2}\epsilon_{\mu\nu\rho}F^{\nu\rho}=0. \tag{3.115}$$

Comparing this to (3.111), we see that the condition (3.112) on θ_{YM} is the condition under which the boundary conditions derived from the action are compatible with supersymmetry.[11]

The condition (3.112) is essentially one that we have already seen. In eqn. (3.6), we write the condition for supersymmetry of a D3-brane plus a (p,q)-fivebrane. For an NS-fivebrane, that is for $p=1$, $q=0$, the condition is

$$-\frac{\cos\psi}{\sin\psi}=\frac{\mathrm{Im}\,\tau}{\mathrm{Re}\,\tau}=\frac{4\pi/e^2}{\theta_{YM}/2\pi}, \tag{3.116}$$

[11] The action (3.114) also allows Dirichlet boundary conditions, in which the connection and its variation vanish on the boundary, but this is not compatible with supersymmetry.

and this is equivalent to (3.112). So this is really the expected condition for the supersymmetry of a D3-brane ending on an NS5-brane.

The physics should be unchanged if we replace θ_{YM} by $\theta_{YM} + 2\pi q$ and replace the NS5-brane with a $(1, q)$-fivebrane. Shifting θ_{YM} by $2\pi q$ adds to the action a bulk "topological" term with that coefficient. To ensure that the results are independent of q, it must be that replacing the NS5-brane with a $(1, q)$-fivebrane has the effect of adding to the effective action for D3-branes ending on the fivebrane a boundary Chern-Simons coupling with coefficient $-q$.

3.5.2 Including Hypermultiplets

Now we want to add three-dimensional hypermultiplets supported at $x^3 = 0$ and transforming in some pseudoreal representation of the gauge group. As in section 3.2, we represent them by $\mathcal{N} = 1$ superfields $\mathcal{Q}^I_A = Q^I_A + \theta^\alpha \lambda^I_{\alpha\, A} + \dots$; moreover, we construct $\mathcal{N} = 1$ couplings that have $SU(2)_d$ symmetry acting on these superfields and adjust those couplings so that the symmetry group is enlarged to $SU(2) \times SU(2)$, with one factor acting on Q and one on λ. How this $SU(2) \times SU(2)$ relates to the $SO(3)_X \times SO(3)_Y$ symmetry that rotates the fields $\vec{X}$ and $\vec{Y}$ of the bulk theory will become clear momentarily.

Once we include hypermultiplets, it is important to consider the case that the gauge group G is a product of simple factors G_i. (The non-trivial examples generally have more than one factor.) Each Lie algebra $\mathfrak{g}_i$ has a quadratic form $(\ ,\ )_i$, which we denote by $(a, b)_i = -\mathrm{Tr}\, ab$. Each factor has its own gauge coupling e_i, its own theta-angle $\theta_{YM,i}$, and its own supersymmetry angle ψ_i. They each obey (3.112)

$$\frac{\theta_{YM,i}}{2\pi} = -\frac{4\pi}{e_i^2} \frac{\sin \psi_i}{\cos \psi_i}. \tag{3.117}$$

The derivation of this formula (either by canceling a σ_A^2 coupling or by considering the boundary condition obey by F) is unaffected by the existence of hypermultiplets. In addition, if we set $D = 1/e^2 \sin 2\psi$, it will turn out that all factors of the gauge group have the same value of D.

Each factor G_i of the gauge group will be localized either at $x^3 \geq 0$ or at $x^3 \leq 0$. Until it is necessary to combine the different semisimple factors of the gauge group, we will proceed as if there were just one factor G.

The Hypermultiplet Action

Now we consider the part of the action that involves the hypermultiplets. The gauge-covariant kinetic energy of $\mathcal{Q}$ is familiar:

$$-\frac{1}{2}\int \mathrm{d}^2\theta(\mathcal{D}_\alpha \mathcal{Q}^I_A)^2 = \frac{1}{2}\epsilon^{AB}\left(-\omega_{IJ}D_\mu Q^I_A D^\mu Q^J_B + \omega_{IJ}\lambda^{I\alpha}_A(i\not{D})_\alpha^\beta\lambda^J_{B\beta} + \omega_{IJ}F^I_{QA}F^J_{QB} + 2\lambda^{\alpha I}_A\tau^m_{IJ}\sigma_{Am\alpha}Q^J_B\right). \tag{3.118}$$

As in the purely three-dimensional problem, since $\mathcal{Q}$ has dimension 1/2, a general quartic superpotential for the matter theory will preserve superconformal invariance at least classically:

$$\int \mathrm{d}^2\theta W_4(\mathcal{Q}). \tag{3.119}$$

Unlike the purely three-dimensional case, we now have dimension 1 bulk superfields $\mathcal{X}$ and $\mathcal{Y}$, so we can preserve conformal symmetry with a superpotential coupling $\mathcal{X}\mathcal{Q}^2$ or $\mathcal{Y}\mathcal{Q}^2$, where here of course $\mathcal{X}$ and $\mathcal{Y}$ are evaluated at $x^3=0$. Actually, in the unperturbed problem, the boundary condition on $\mathcal{Y}$ was simply $\mathcal{Y}=0$, so a boundary coupling involving $\mathcal{Y}$ does not add anything.[12] As we will see, R-symmetry requires a $\mathcal{X}\mathcal{Q}^2$ term in the superpotential. It turns out that $\mathcal{X}$ should couple precisely to the moment map superfield $\mathcal{M}^m_{AB} = \mathcal{Q}^I_A\mathcal{Q}^J_B\tau^m_{IJ}$. In order to facilitate the calculations, we will replace in the remainder of this section the vector indices with symmetric pairs of doublet indices: for example the $\mathcal{X}\mathcal{Q}^2$ interaction term is

$$c\int \mathrm{d}^2\theta \mathcal{X}^{AB}_m \mathcal{Q}^I_A\mathcal{Q}^J_B\tau^m_{IJ} = c\tau^m_{IJ}(F^{AB}_{xm}Q^I_AQ^J_B + 2X^{AB}_m F^I_{qA}Q^J_B + 2\rho^{AB\alpha}_{1m}\lambda^I_{A\alpha}Q^J_B + X^{AB}_m\lambda^I_A\lambda^J_B) \tag{3.120}$$

Basic formulae about this replacement are collected in appendix D. An important one is that $X^{AB}X_{AB} = -2X^aX^a$. The $\delta(x^3)$ term in the $\mathcal{X}$ gradient of the superpotential becomes

$$\frac{1}{e^2}Y^m_{AB} + c\tau^m_{IJ}Q^I_AQ^J_B. \tag{3.121}$$

The vanishing of the $\delta(x^3)^2$ term in the action now tells us that the Dirichlet boundary condition on Y must be modified to

$$Y^{ma} = -ce^2\tau^m_{IJ}(Q^I\sigma^aQ^J). \tag{3.122}$$

Again, this boundary condition will imply a set of boundary conditions for the other fields. By $\mathcal{N}=1$ supersymmetry,

$$\rho^m_{\alpha AB} = -ce^2\tau^m_{IJ}Q^I_{(A}\lambda^J_{B)\alpha}. \tag{3.123}$$

Extending this by R-symmetry, we get the most interesting relation:

$$\cos\psi\sigma_{3\alpha} + \sin\psi\sigma_{A\alpha} = -ce^2\tau^m_{IJ}Q^{IA}\lambda^J_{A\alpha}. \tag{3.124}$$

[12] In the combined system, the boundary condition $\mathcal{Y}=0$ is modified to $\mathcal{Y}\sim\mathcal{Q}^2$, as we see momentarily. Still this means that a $\mathcal{Y}\mathcal{Q}^2$ superpotential interaction is equivalent to a modification of the quartic superpotential $W_4(\mathcal{Q})$.

One further application of $\mathcal{N}=1$ supersymmetry gives another interesting relation:

$$F_{3\mu} = \tan\psi \frac{1}{2}\epsilon_{\mu\nu\rho}F^{\nu\rho} - \frac{ce^2}{\cos\psi}J^\mu. \tag{3.125}$$

(Appendix B is useful to understand the precise normalization.)

This should be compared to free boundary conditions on a gauge field with a theta-angle and a boundary coupling:

$$I = \int_M \mathrm{d}^4x \left(\frac{1}{2e^2}F_{IJ}F^{IJ} - \frac{\theta_{YM}}{32\pi^2}\epsilon^{IJKL}\mathrm{Tr}F_{IJ}F_{KL}\right) + \int_{\partial M} I'. \tag{3.126}$$

Requiring that I should be stationary with no restriction on the variation of A at the boundary, we get a boundary condition on A that coincides with (3.125), if the current of the boundary fields is defined as usual by $J^\mu = \delta I'/\delta A_\mu$ and if in addition

$$c = \frac{1}{2}\cos\psi. \tag{3.127}$$

The same restriction on c will appear in a moment from the R-symmetry analysis. The boundary condition on Y can be written in terms of the moment map

$$Y^{ma} = -\frac{1}{2}e^2\cos\psi\mu^{ma}. \tag{3.128}$$

For this equation to be R-symmetric, the $SU(2)$ factor in the R-symmetry group that acts on Y must be the same as the one that acts on Q. So we aim for a construction in which one factor in the $SU(2)\times SU(2)$ R-symmetry group rotates Q and $\vec{Y}$, while the other rotates $\vec{X}$ and λ.

The non-R-symmetric bosonic boundary terms of eqn. (3.104), namely

$$-\epsilon_{abc}\delta(x_3)\mathrm{Tr}\left(\cos\psi\frac{1}{e^2}Y^a[X^b,X^c] + \sin\psi\frac{1}{e^2}X^a[Y^b,Y^c]\right), \tag{3.129}$$

were dismissed in section 3.5.1 because of the boundary condition $\vec{Y}=0$. These terms are now equivalent to X^2Q^2 and XQ^4 boundary couplings, which will in general break R-symmetry. The existence of separate $SU(2)$ groups rotating $\vec{X}$ and Q strongly constrains X^2Q^2 interactions and means that XQ^4 interactions should be absent. We can rewrite the preceding formula in terms of doublet indices

$$-i\delta(x_3)\mathrm{Tr}\left(\cos\psi\frac{1}{2e^2}Y^A_B[X^B_C,X^C_A] + \sin\psi\frac{1}{2e^2}X^A_B[Y^B_C,Y^C_A]\right). \tag{3.130}$$

There are further non-R-symmetric terms arising from the superpotential. Its derivative with respect to Q is

$$2c\tau^m_{IJ}X^{AB}_mQ^J_B + \partial^A_IW_4 \tag{3.131}$$

and the square of this gives interactions

$$2c^2\tau^m_{IJ}\tau^n_{KT}X^{AB}_m X^{CD}_n Q^J_B Q^T_D\omega_{IK}\epsilon_{AC} + 2c\tau^m_{IJ}X^{AB}_m Q^J_B\partial^I_A W_4. \tag{3.132}$$

Now we have several kind of terms to play against each other. If the terms proportional to X^aX^b are symmetrized in a and b, the result is actually proportional to $\vec{X}\cdot\vec{X}$, and thus is R-symmetric, that is, invariant under separate rotations of X and Q. However, the part antisymmetric in a and b, which contracts $\epsilon_{abc}X^bX^c$ with an expression bilinear in Q, is non-R-symmetric. It is

$$c^2\left(\tau^m_{IJ}\omega_{IK}\tau^n_{KT} - \tau^n_{IJ}\omega_{IK}\tau^m_{KT}\right)X^{AB}_m X^{CD}_n Q^J_B Q^T_D\epsilon_{AC} = c^2 f^{mnp}X^{AB}_m X^{CD}_n\epsilon_{AC}Q^J_B\tau_{pJT}Q^T_D. \tag{3.133}$$

On the other hand, we can apply the Y boundary condition to the first term in (3.130) and replace the trace $\mathrm{Tr}Y[X,X]$ with an explicit sum over the gauge group structure constants:

$$\frac{1}{2}\cos\psi c f^{mnp}X^B_{mC}X^C_{nA}\tau_{pIJ}Q^{IA}Q^J_B. \tag{3.134}$$

The $QQXX$ terms cancel against each other due to the boundary conditions on Y if $c = \frac{1}{2}\cos\psi$. We will deal with the XQ^4 interaction momentarily.

The Yukawa Couplings

Now we come to what is in a sense the main point: for the configuration to be supersymmetric, we require just the same condition on the gauge group and hypermultiplet representation as in section 3.2. As before, this result will come from ensuring R-symmetry of the "Yukawa couplings" $Q^2\lambda^2$. This is the only point in the derivation at which we sum over all simple factors G_i in the gauge group.

There are two sources of $Q^2\lambda^2$ couplings. One is (3.113); the second comes from the R-symmetrization of the coupling

$$\cos\psi\rho^{AB\alpha}_{1m}\lambda^I_{A\alpha}\tau^m_{IJ}Q^J_B. \tag{3.135}$$

This requires as usual a term

$$\cos\psi(\cos\psi\sigma_{Am} - \sin\psi\sigma_{3m})\lambda^I_{A\alpha}\tau^m_{IJ}Q^{JB} \tag{3.136}$$

but only

$$\lambda^I_A\tau^m_{IJ}Q^{JA}\sigma_{Am} \tag{3.137}$$

is present in the Lagrangian. As a result, after completing the ρ coupling to an R-symmetric coupling, one is left with

$$\sin\psi(\sin\psi\sigma_{Am} + \cos\psi\sigma_{3m})\lambda^I_{A\alpha}\tau^m_{IJ}Q^{JB} \tag{3.138}$$

If we start from this term and from (3.113) and apply the fermion boundary conditions (3.124), we get the following key boundary Yukawa coupling:

$$\pi Q^I_A Q^J_B \epsilon^{\alpha\beta} \lambda^K_{\alpha\dot{C}} \lambda^S_{\beta\dot{D}} \epsilon^{A\dot{C}} \epsilon^{B\dot{D}} \tau^m_{IK} \tau^n_{JS} \widetilde{k}_{mn}. \tag{3.139}$$

The interaction (3.139) involves hypermultiplet fields only, and receives contributions from every factor G_i. In this formula, $\widetilde{k}^{mn}$ is a quadratic form on the Lie algebra of G that is defined as follows. On $\mathfrak{g}_i$, the quadratic form equals $\pm 8\pi(\ ,\ )_i/e_i^2 \sin 2\psi_i$, where the sign is $+$ or $-$ according to whether the group G_i is supported for $x^3 < 0$ or for $x^3 > 0$.

Eqn. (3.139) is not R-symmetric, but it is identical in form to the first term in eqn. (3.21) – the term which in the purely three-dimensional derivation came from integrating out the auxiliary field χ. So the cure is the same as in section 3.2. After picking the same superpotential $\mathcal{W}_4 = \frac{\pi}{6}\widetilde{k}_{mn}\mu^{m\,AB}\mu^n{}_{AB}$ as in the purely three-dimensional case, we can combine two kinds of Yukawa couplings into an R-symmetric combination. In doing so, we have to obey the same constraint on the matter and gauge content as for the Chern-Simons theory. So G must be the bosonic part of a supergroup $\widehat{G}$, whose Lie algebra has an invariant, nondegenerate quadratic form whose restriction to $\mathfrak{g}$ is $\widetilde{k}$.

Application To The D3-NS5 System

Let us specialize this to the D3-NS5 system, with N D3-branes on one side of an NS5-brane, and M on the other. The supergroup is $U(N|M)$, and the gauge group is $G = U(N) \times U(M)$. The usual invariant quadratic form k on $\mathfrak{g}$ is equal to the trace Tr on one summand of $\mathfrak{g}$ and $-$Tr on the other. The form $\widetilde{k}$ found above must be a multiple of this (since it must be the restriction to $\mathfrak{g}$ of an invariant quadratic form on the super Lie algebra). Hence, writing e_1, e_2 and ψ_1, ψ_2 for the values in the two factors, we have

$$e_1^2 \sin 2\psi_1 = e_2^2 \sin 2\psi_2. \tag{3.140}$$

This result along with (3.112) has a simple interpretation. It says that the points $(e_1, \theta_{YM,1}, \psi_1)$ and $(e_2, \theta_{YM,2}, \psi_2)$ obey a relation of the specific form $\tau = 4\pi D(\exp(2i\psi) - 1)$. As ψ varies, this defines a semicircle in the upper half plane whose rightmost intersection with the real axis is at $\tau = 0$. We explained in section 3.1.4 that a Janus configuration of precisely that type preserves the same supersymmetry as an NS5-brane. Hence, there should be a low energy supersymmetric action describing an NS5-brane interacting with such a Janus configuration. This is what we have found, at least for the case that the couplings change only by jumping in crossing the fivebrane. The extension to the general case is immediate, since, as in section 3.5.1, even if we let e^2 and θ_{YM} vary, the boundary terms do not depend on their derivatives.

Actually, as explained in section 3.1.3, the existence of the general Janus configuration can be inferred from the properties of the D3-NS5 system. We simply consider, as in fig. 2, a system of D3-branes interacting with NS5-branes located at different values of x^3, with constant couplings e_i and $\theta_{YM,i}$ and supersymmetry parameters ψ_i in between the NS5-branes. We take the number of D3-branes to everywhere equal N. We describe each interface by the above construction, with jumps in couplings that are constrained by (3.112) and by the fact that $D = 1/e_i^2 \sin 2\psi_i$ must be constant. Thus the couplings all take values in the usual semicircle.

Finally, we remove the NS5-branes by displacing them in the $x^7 - x^8 - x^9$ directions. This causes the various $U(N)$ gauge groups (in the half-spaces and slabs separated by NS5-branes) to recombine into a single four-dimensional gauge group. The couplings, however, jump in a discrete version of the Janus configuration, which can approximate a continuous Janus configuration when the number of NS5-branes is very large.

The process of displacing the NS5-branes so that they do not meet the D3-branes corresponds in field theory to giving expectation values to the hypermultiplet fields Q. (This is familiar in the absence of the theta-angle.) As we explained in analyzing eqn. (3.35), setting the potential energy to zero requires that the matrices $Q_A Q^\dagger{}_B$ commute, and likewise the matrices $Q^\dagger{}_B Q_A$. Generic expectation values for these matrices break the $U(N) \times U(N)$ symmetry to a diagonal $U(N)$ or a subgroup thereof.

If we symmetrize in A and B, the matrices $QQ^\dagger{}_{(AB)}$ and $Q^\dagger Q_{(AB)}$ become the moment maps for the two factors of the gauge group, and according to (3.128), they are proportional to $\vec{Y}$ at $x^3 = 0$. The matrices $QQ^\dagger{}_{(AB)}$ and $Q^\dagger Q_{(AB)}$ have the same eigenvalues, so the boundaries values of $\vec{Y}$ in one $U(N)$ group are conjugate to those in the other, consistent with the claim that the symmetry breaking combines the two $\vec{Y}$ fields into a single such field of a single $U(N)$ gauge symmetry. If the matrices $QQ^\dagger{}_{(AB)}$ and $Q^\dagger Q_{(AB)}$ are (large) multiples of the identity, the $U(N)$ gauge symmetry is unbroken, the NS5-branes decouple, and we reduce to a discrete Janus configuration.

The XQ^4 Terms

We still have to check the vanishing of the XQ^4 term in the action. One contribution from the XYY bulk total derivative is simply

$$-i\frac{e^2}{4}\sin\psi\cos^2\psi f^{mnp} X^A_{mB}\mu^B_{C\,n}\mu^C_{A\,p}. \tag{3.141}$$

The second comes from the $\mathcal{Q}$ derivative of the superpotential and is more complex:

$$\frac{2\pi}{3}\widetilde{k}_{mn}\cos\psi\mu^{ABm}Q^I_A(\tau^m\tau^n)_{IJ}Q^{JC}X_{BCn}. \tag{3.142}$$

These terms better cancel each other out. We need to rearrange the second term quite a bit. We can use an antisymmetrization of the AC upper indices first to transform it to

$$\pi\widetilde{k}_{mn}\cos\psi\left(\mu^{ABm}Q^I_A(\tau^m\tau^n)_{IJ}Q^{JC}X_{BCn}-\frac{1}{3}\mu^{BCm}Q^I_A(\tau^m\tau^n)_{IJ}Q^{JA}X_{BCn}+\frac{1}{3}\mu^{ABm}Q^{IC}(\tau^m\tau^n)_{IJ}Q^J_AX_{BCn}\right) \tag{3.143}$$

Now we can use the (3.23) identity to combine the last two terms into

$$\pi\widetilde{k}_{mn}\cos\psi\left(\mu^{ABm}Q^I_A(\tau^m\tau^n)_{IJ}Q^{JC}X_{BCn}+\mu^{ABm}Q^{IC}(\tau^m\tau^n)_{IJ}Q^J_AX_{BCn}\right). \tag{3.144}$$

Finally, these two terms differ only by the order of τ^m and τ^n, and one can use the Jacobi identity on the gauge generators to finally recast it in the same form as 3.141, but with opposite sign

$$\pi\widetilde{k}_{mn}\cos\psi\mu^{ABm}Q^I_A\tau^k_{IJ}Q^{JC}X^n_{BC}f_{mnk}. \tag{3.145}$$

Coupling to General CFT

As in the purely three-dimensional case of section 3.2, it is natural to express the various ingredients in this construction in terms of the current supermultiplet, with an eye towards a generalization involving a generic supersymmetric model that satisfies the fundamental identity. For example, the boundary coupling to X can be expressed in terms of the $\mathcal{N}=1$ supermultiplet $\mathcal{M}$ whose lowest term is the moment map:

$$\frac{1}{2}\cos\psi\int \mathrm{d}^2\theta\mathcal{X}^{AB}_m\mathcal{M}^m_{AB}. \tag{3.146}$$

The boundary conditions on the bulk fields have the various members of the $\mathcal{N}=4$ current supermultiplet on the right hand side, and are immediately generalized to the hyper-Kahler sigma model. The proof of R-symmetry of the Yukawa couplings is identical to the Chern-Simons case, and leads one to hyper-Kahler manifolds with moment maps satisfying the fundamental identity. The only new ingredient is to prove that the non-R-symmetric terms of the form X^2Q^2 and XQ^4 do cancel as a consequence of the fundamental identity as well. As the superpotential is given in terms of the moment map superfield, the bosonic terms will involve the gradient squared of moment maps. On the other hand the terms which come from the total derivatives $\partial_3\mathrm{Tr}XXY$ and $\partial_3\mathrm{Tr}XYY$ involve the moment maps directly. To be able to compare the two terms, we will need a simple but useful relation from symplectic geometry between the Poisson bracket of moment maps and the moment map of the Lie bracket of the corresponding vector fields:

$$i_{V^n}\left(i_{V^m}(\omega_{AB})\right)=f_{mnp}\mu^p_{AB} \tag{3.147}$$

Here is a useful consequence of this relation:

$$\frac{1}{2}(d\mu^{[m}_{AB},d\mu^{n]}_{CD})=\frac{1}{2}(i_{V^{[m}}\omega_{AB},i_{V^{n]}}\omega_{CD})=\epsilon_{BC}i_{V^n}\left(i_{V^m}(\omega_{AD})\right)+sym=\epsilon_{BC}f^{mn}_p\mu^p_{AD}+sym \tag{3.148}$$

We used the fact that the three complex structures have the same algebra as the unit quaternions, so that up to an appropriate constant $\omega^1_{ij}\omega^2_{kt}g^{jk} = \omega^3_{it}$ and $\omega^1_{ij}\omega^1_{kt}g^{jk} = g_{it}$ (here i, j, k, t are indices in the tangent or cotangent bundle). In particular it is also true that

$$\frac{1}{2}(d\mu^{(m}_{AB}, d\mu^{n)}_{CD}) = \frac{1}{2}(i_{V^{(m}}\omega_{AB}, i_{V^{n)}}\omega_{CD}) = \epsilon_{BC}\epsilon_{AD}(V^n, V^m) + sym \tag{3.149}$$

Let us put these two relations to work to cancel the non R-symmetric $X^2\mu$ and $X\mu^2$ terms in the Lagrangian. In particular 3.148 will play the role that the Jacobi identity for τ played in the free hypermultiplet case. The $XX\mu$ non-R-symmetric term from the superpotential is

$$\frac{1}{8}\cos^2\psi X^{AB}_m X^{CD}_n (d\mu^m_{AB}, d\mu^n_{CD}) \tag{3.150}$$

More precisely the part symmetric in (AB) and (CD) is proportional again to $X^a X^a$ by 3.149 and is R-symmetric, while the antisymmetric part is

$$\frac{1}{2}\cos^2\psi X^{AB}_m X^{CD}_n \epsilon_{BC} f^{mn}_p \mu^p_{AD} \tag{3.151}$$

by 3.148 and cancels against the TrXXY total derivative. The analysis of the terms linear in X proceeds quite smoothly as well: we start with

$$\frac{\pi}{6}\cos\psi \widetilde{k}_{mp}\mu^p_{AB} X^{CD}_n (d\mu^m_{AB}, d\mu^n_{CD}) \tag{3.152}$$

and proceed in complete parallelism to the free field computation.

The quiver construction of section 3.2.3 gives examples of three-dimensional superconformal field theories that obey the fundamental identity, so that the above analysis is applicable. Actually, we can now motivate this quiver construction. In fig. 2 of section 3.2.3, the slabs between two NS5-branes are macroscopically only three-dimensional (as the x^3 coordinate is bounded between two branes), so at low energies one can use an effective three-dimensional description of the gauge fields that live in the slabs. In this description, the three-dimensional gauge fields in the slabs have no Chern-Simons couplings, since the contributions to those couplings cancel at the two ends of the slab. Hence these gauge fields only have ordinary F^2 kinetic energy. In the limit that the slabs are thin, the three-dimensional gauge fields become strongly coupled. We can integrate them out of the low energy analysis by taking a hyper-Kahler quotient of the hypermultiplets that they couple to. This leads to the quiver construction of section 3.2.3. The theories that arise from the quiver construction must obey the fundamental identity; indeed, we know from the analysis of the D3-NS5 system that the brane configuration of fig. 2 can be coupled to bulk gauge fields with $\theta_{YM} \neq 0$ (even before taking the limit that the slabs become thin). We verified the fundamental identity directly in section 3.2.3.

A Some useful relations about spin indices

The obvious identity

$$A_{[\alpha}B_{\beta]} = -C_{\alpha\beta}A^{\gamma}B_{\gamma} \tag{A.1}$$

will be often useful, along with the similar formula

$$A^{\alpha}B_{\beta} - A_{\beta}B^{\alpha} = \delta^{\alpha}_{\beta}A^{\gamma}B_{\gamma}. \tag{A.2}$$

A vector is represented in spinor notation as a matrix

$$\not{V}_{\alpha\beta} = \begin{pmatrix} V_0 + V_1 & V_2 \\ V_2 & V_0 - V_1 \end{pmatrix}. \tag{A.3}$$

The norm is

$$\not{V}^{\alpha\beta}\not{W}_{\alpha\beta} = 2V^{\mu}W_{\mu}. \tag{A.4}$$

in signature $-++$. The exterior product is computed by

$$\not{V}_{\alpha\beta}\not{W}_{\gamma\delta}C^{\beta\gamma} + \not{V}_{\delta\beta}\not{W}_{\gamma\alpha}C^{\beta\gamma} = 2i\not{S}_{\alpha\delta} \qquad S = V \wedge W \tag{A.5}$$

Moreover

$$\not{V}_{\alpha\beta}\not{W}_{\gamma\delta}\not{S}_{\rho\lambda}C^{\beta\gamma}C^{\delta\rho}C^{\lambda\alpha} = -2i\epsilon^{\mu\nu\sigma}V_{\mu}W_{\nu}S_{\sigma} \tag{A.6}$$

Similar formulae are valid for $SU(2)$ indices, raised and lowered by the conventional alternating tensor $\epsilon_{12} = \epsilon^{12} = 1$. A vector is represented in spinor notation as

$$\not{V}_{AB} = \begin{pmatrix} iV_2 + V_1 & V_3 \\ V_3 & iV_2 - V_1 \end{pmatrix}. \tag{A.7}$$

The norm is

$$\not{V}^{AB}\not{W}_{AB} = -2V^{\mu}W_{\mu}. \tag{A.8}$$

The exterior product is computed by

$$\not{V}_{AB}\not{W}_{CD}\epsilon^{BC} + \not{V}_{DB}\not{W}_{CA}\epsilon^{BC} = 2i\not{S}_{AD} \qquad S = V \wedge W \tag{A.9}$$

Moreover

$$\not{V}_{AB}\not{W}_{CD}\not{S}_{EF}\epsilon^{BC}\epsilon^{DF}\epsilon^{FA} = 2i\epsilon^{abc}V_aW_bS_c \tag{A.10}$$

B Some Miscellanea About $\mathcal{N} = 1$

The content of this appendix is mostly trivial, but it is a useful gymnastic in preparation to the next appendix, where the closure of the $\mathcal{N} = 4$ SUSY algebra is explicitly checked for our Chern-Simons theory. Let us do a bit of $\mathcal{N} = 1$ SUSY to check the superfield lagrangians. Basic real superfield

$$\begin{aligned} \delta\phi &= \varepsilon^\alpha \psi_\alpha \\ \delta\psi_\alpha &= \varepsilon^\beta i\partial\!\!\!/_{\alpha\beta}\phi - \varepsilon_\alpha F \\ \delta F &= \varepsilon^\alpha i\partial\!\!\!/_\alpha^\beta \psi_\beta \end{aligned} \tag{B.1}$$

$$\begin{aligned} \delta^2\phi &= \varepsilon^\alpha \varepsilon^\beta i\partial\!\!\!/_{\alpha\beta}\phi \\ \delta^2\psi_\alpha &= \varepsilon^\beta \varepsilon^\gamma i\partial\!\!\!/_{\alpha\beta}\psi_\gamma - \varepsilon_\alpha \varepsilon^\beta i\partial\!\!\!/_\beta^\gamma \psi_\gamma = \varepsilon^\beta \varepsilon^\gamma i\partial\!\!\!/_{\gamma\beta}\psi_\alpha \\ \delta^2 F &= \varepsilon^\alpha \varepsilon^\gamma i\partial\!\!\!/_\alpha^\beta i\partial\!\!\!/_{\gamma\beta}\phi - \varepsilon^\alpha \varepsilon_\beta i\partial\!\!\!/_\alpha^\beta F = \varepsilon^\alpha \varepsilon^\beta i\partial\!\!\!/_{\alpha\beta} F \end{aligned} \tag{B.2}$$

Kinetic terms:

$$\begin{aligned} \delta\left(\frac{1}{2}F^2\right) &= \varepsilon^\alpha F i\partial\!\!\!/_\alpha^\beta \psi_\beta \\ \delta\left(\frac{1}{2}\psi^\alpha i\partial\!\!\!/_\alpha^\beta \psi_\beta\right) &= \varepsilon^\gamma i\partial\!\!\!/_\gamma^\alpha \phi i\partial\!\!\!/_\alpha^\beta \psi_\beta - \varepsilon^\alpha F i\partial\!\!\!/_\alpha^\beta \psi_\beta = -\frac{1}{2}\varepsilon^\gamma i\partial\!\!\!/_\alpha^\beta i\partial\!\!\!/_\beta^\alpha \phi\psi_\gamma - \varepsilon^\alpha F i\partial\!\!\!/_\alpha^\beta \psi_\beta \\ \delta\left(-\frac{1}{4} i\partial_\beta^\alpha \phi i\partial_\alpha^\beta \phi\right) &= -\frac{1}{2}\varepsilon^\gamma i\partial_\beta^\alpha \phi i\partial_\alpha^\beta \psi_\gamma = \frac{1}{2}\varepsilon^\gamma i\partial\!\!\!/_\alpha^\beta i\partial\!\!\!/_\beta^\alpha \phi\psi_\gamma \end{aligned} \tag{B.3}$$

Superpotentials:

$$\begin{aligned} \delta\left(W(\phi)' F\right) &= W'' \varepsilon^\alpha \psi_\alpha F + W' \varepsilon^\alpha i\partial\!\!\!/_\alpha^\beta \psi_\beta \\ \delta\left(W(\phi)'' \frac{1}{2}\psi^\alpha \psi_\alpha\right) &= \frac{1}{2} W''' \varepsilon^\beta \psi_\beta \psi^\alpha \psi_\alpha + W'' \varepsilon^\beta i\partial\!\!\!/_\beta^\alpha \phi\psi_\alpha - W'' \varepsilon^\alpha F \psi_\alpha \end{aligned} \tag{B.4}$$

For the gauge multiplet

$$\begin{aligned} \delta A\!\!\!/_{\alpha\beta} &= \varepsilon_\alpha \chi_\beta + \varepsilon_\beta \chi_\alpha \\ \delta\chi_\alpha &= \varepsilon^\beta f_{\alpha\beta} \end{aligned} \tag{B.5}$$

$$
\begin{aligned}
\delta^2 \not{A}_{\alpha\beta} &= \varepsilon_\alpha \varepsilon^\gamma f_{\beta\gamma} + \varepsilon_\beta \varepsilon^\gamma f_{\alpha\gamma} \\
\delta^2 \chi_\alpha &= \varepsilon^\beta \delta f_{\alpha\beta}
\end{aligned} \tag{B.6}
$$

We need $\delta^2 \not{A}_{\alpha\beta} = \varepsilon^\gamma \varepsilon^\delta i \not{F}_{\gamma\delta;\alpha\beta}$ and we learn

$$
i\not{F}_{\gamma\delta;\alpha\beta} = \frac{1}{2}\epsilon_{\delta\alpha} f_{\beta\gamma} + \frac{1}{2}\epsilon_{\gamma\alpha} f_{\beta\delta} + \frac{1}{2}\epsilon_{\delta\beta} f_{\alpha\gamma} + \frac{1}{2}\epsilon_{\gamma\beta} f_{\alpha\delta} \tag{B.7}
$$

and

$$
f_{\alpha\beta} = \frac{1}{2} i \not{F}^{\gamma}_{\alpha\gamma;\beta} \tag{B.8}
$$

Hence

$$
\varepsilon^\beta \delta f_{\alpha\beta} = \frac{1}{2}\left(\varepsilon^\beta \varepsilon^\gamma i\not{\partial}_{\alpha\gamma}\chi_\beta + \varepsilon^\beta \varepsilon_\beta i\not{\partial}_{\alpha\gamma}\chi^\gamma - \varepsilon^\beta \varepsilon_\alpha i\not{\partial}^{\gamma}_{\beta}\chi_\gamma - \varepsilon^\beta \varepsilon_\gamma i\not{\partial}^{\gamma}_{\beta}\chi_\alpha\right) \tag{B.9}
$$

and works.

Kinetic terms

$$
\begin{aligned}
&\delta \frac{1}{2}\chi^\alpha i\not{\partial}^{\beta}_{\alpha}\chi_\beta = \varepsilon^\gamma f^{\alpha}_{\gamma} i\not{\partial}^{\beta}_{\alpha}\chi_\beta \\
&\delta - \frac{1}{4} f^{\alpha\beta} f_{\alpha\beta} = -\frac{1}{2}\left(f^{\alpha\beta}\varepsilon^\gamma i\not{\partial}_{\alpha\gamma}\chi_\beta + f^{\alpha\beta}\varepsilon_\beta i\not{\partial}_{\alpha\gamma}\chi^\gamma\right) = -\frac{1}{2}\left(f^{\alpha\gamma}\varepsilon^\beta i\not{\partial}_{\alpha\gamma}\chi_\beta + 2 f^{\alpha\beta}\varepsilon_\beta i\not{\partial}_{\alpha\gamma}\chi^\gamma\right)
\end{aligned} \tag{B.10}
$$

Remember $\not{\partial}^{\alpha\beta} f_{\alpha\beta} = 0$. More conventionally,

$$
-\frac{1}{4} f^{\alpha\beta} f_{\alpha\beta} = \frac{1}{16}\not{F}^{\alpha;\beta\gamma}_{\gamma}\not{F}^{\delta}_{\alpha\delta;\beta} = \frac{1}{16}\not{F}^{\alpha;\beta\delta}_{\gamma}\not{F}^{\gamma}_{\alpha\delta;\beta} + \frac{1}{16}\not{F}^{\alpha\gamma;\beta\delta}\not{F}_{\alpha\gamma;\beta\delta} = \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \tag{B.11}
$$

The first term drops by symmetries.

Chern Simons term

$$
\begin{aligned}
\delta \frac{1}{2}\chi^\alpha \chi_\alpha &= \varepsilon^\gamma f^{\alpha}_{\gamma}\chi_\alpha \\
\delta \frac{1}{4}\not{A}_{\alpha\beta} f^{\alpha\beta} &= f^{\alpha\beta}\varepsilon_\alpha \chi_\beta
\end{aligned} \tag{B.12}
$$

More conventionally,

$$
\frac{1}{4}\not{A}^{\alpha\beta} f_{\alpha\beta} = \frac{1}{8}\not{A}^{\alpha\beta}\not{F}^{\gamma}_{\alpha\gamma;\beta} = \frac{1}{8} C^{\alpha\delta} C^{\beta\sigma} C^{\gamma\tau}\not{A}_{\delta\sigma}\not{F}_{\alpha\gamma;\beta\tau} = \frac{1}{4}\epsilon^{\mu\nu\rho} A_\mu F_{\nu\rho} \tag{B.13}
$$

Consider some real multiplets in a real representation of the gauge group, with covariant derivative $iD_\mu = i\partial_\mu \phi^i + A_\mu T^i_j \phi^j$. The coupling to gauge fields is

$$\frac{1}{2}\not{J}^{\alpha\beta}\not{A}_{\alpha\beta} = \frac{1}{2}\psi^\alpha \not{A}^\beta_\alpha T\psi_\beta - \frac{1}{2}\not{A}^\alpha_\beta T\phi i\not{\partial}^\beta_\alpha \phi \tag{B.14}$$

The current

$$-\psi^\alpha T\psi^\beta - \phi T i\not{\partial}^{\alpha\beta}\phi \tag{B.15}$$

Notice that if $j^\alpha = \phi T\psi^\alpha$

$$\delta j^\alpha = \varepsilon^\beta \psi_\beta T\psi^\alpha + \varepsilon^\beta \phi T i\not{\partial}^\alpha_\beta \phi + \varepsilon^\alpha \phi TF = \varepsilon_\beta \not{J}^{\alpha\beta} + \varepsilon^\alpha \phi TF \tag{B.16}$$

Hence

$$\begin{aligned} \delta\frac{1}{2}\not{J}^{\alpha\beta}\not{A}_{\alpha\beta} &= \not{J}^{\alpha\beta}\varepsilon_\alpha \chi_\beta + \cdots \\ \delta - j^\alpha \chi_\alpha &= -\varepsilon_\beta \not{J}^{\alpha\beta}\chi_\alpha \end{aligned} \tag{B.17}$$

C Closure of the Chern-Simons Supersymmetry Algebra

The supersymmetry transformations are

$$\begin{aligned} \delta Q^I_A &= \varepsilon^{\dot{B}\alpha}_A \lambda^I_{\dot{B}\alpha} \\ \delta\lambda^I_{\dot{A}\alpha} &= \varepsilon^{B\beta}_{\dot{A}} i\not{D}_{\alpha\beta} Q^I_B + \frac{1}{3}\varepsilon^B_{\dot{A}\alpha} T^{mI}_J Q^{JC} Q^K_C \tau^n_{KT} Q^T_B k_{mn} \\ \delta\not{A}_{m\alpha\beta} &= k_{mn}\varepsilon^{A\dot{B}}_{(\alpha}\lambda^I_{\beta)\dot{B}}\tau^n_{IJ} Q^J_A \end{aligned} \tag{C.1}$$

For readability, we will denote $T^{mI}_J Q^{JC}$ as $(TQ)^{IC}$, $Q^K_C \tau^n_{KT} Q^T_B$ as $(Q\tau Q)_{CB}$, $\lambda^I_{\beta)\dot{B}}\tau^n_{IJ}Q^J_A$ as $(\lambda\tau Q)_{\beta\dot{B}A}$ and leave k_{mn} implicit, with a single exception for a formula where two k_{mn} appear.

$$\begin{aligned} \delta Q^I_A &= \varepsilon^{\dot{B}\alpha}_A \lambda^I_{\dot{B}\alpha} \\ \delta\lambda^I_{\dot{A}\alpha} &= \varepsilon^{B\beta}_{\dot{A}} i\not{D}_{\alpha\beta} Q^I_B + \frac{1}{3}\varepsilon^B_{\dot{A}\alpha}(TQ)^{IC}(Q\tau Q)_{CB} \\ \delta\not{A}_{\alpha\beta} &= \varepsilon^{A\dot{B}}_\alpha (\lambda\tau Q)_{\beta\dot{B}A} + \varepsilon^{A\dot{B}}_\beta (\lambda\tau Q)_{\alpha\dot{B}A} \end{aligned} \tag{C.2}$$

The second variation of Q is

$$\delta^2 Q^I_A = \varepsilon^{\dot{B}\alpha}_A \varepsilon^{C\beta}_{\dot{B}} i\not{D}_{\alpha\beta} Q^I_C + \frac{1}{3}\varepsilon^{\dot{B}\alpha}_A \varepsilon^D_{\dot{B}\alpha} (TQ)^{IC}(Q\tau Q)_{CD} k_{mn} \tag{C.3}$$

In the first term the ε^2 part is antisymmetric in A and C and can be rewritten as

$$\frac{1}{2}\varepsilon^{\dot{B}\alpha}_C \varepsilon^{C\beta}_{\dot{B}} i\not{D}_{\alpha\beta} Q^I_A \tag{C.4}$$

and is the conventional gauge covariant translation. The second term needs some rearrangements

$$-\frac{1}{3}\varepsilon^{\dot{B}\alpha}_A \varepsilon^D_{\dot{B}\alpha} (TQ)^I_C (Q\tau Q)^C_D \tag{C.5}$$

Antisymmetrizing on AC gives

$$\frac{1}{3}\varepsilon^{\dot{B}\alpha}_C \varepsilon_{\dot{B}\alpha D} (TQ)^{IC}(Q\tau Q)^D_A - \frac{1}{3}\varepsilon^{\dot{B}\alpha}_C \varepsilon^D_{\dot{B}\alpha} (TQ)^I_A (Q\tau Q)^C_D \tag{C.6}$$

Application of the fundamental identity to the first term by cyclically permuting the JKT indices in $T^{mI}_J \tau^n_{KT} k_{mn}$ gives finally

$$\frac{1}{2}\varepsilon^{\dot{B}\alpha C} \varepsilon^D_{\dot{B}\alpha} \mu^n_{CD} k_{mn} T^{mI}_J Q^J_A. \tag{C.7}$$

This is a gauge transformation by a parameter $\frac{1}{2}\varepsilon^{\dot{B}\alpha C}\varepsilon^D_{\dot{B}\alpha}\mu^n_{CD}k_{mn}$

Let us now look at the fermion supersymmetry transformation:

$$\begin{aligned}\delta^2 \lambda^I_{\dot{A}\alpha} = {} & \varepsilon^{B\beta}_{\dot{A}} \varepsilon^{\dot{C}\gamma}_B i\not{D}_{\alpha\beta} \lambda^I_{\dot{C}\gamma} + \varepsilon^{B\beta}_{\dot{A}} \varepsilon^{C\dot{D}}_\alpha (TQ)^I_B (\lambda\tau Q)_{\beta\dot{D}C} + \varepsilon^{B\alpha}_{\dot{A}} \varepsilon^{C\dot{D}}_\beta (TQ)^I_B (\lambda\tau Q)_{\beta\dot{D}C} + \\ & \frac{1}{3}\varepsilon^B_{\dot{A}\alpha} \varepsilon^{C\dot{D}\beta} (T\lambda)^I_{\dot{D}\beta} (Q\tau Q)_{CB} + \frac{1}{3}\varepsilon^B_{\dot{A}\alpha} \varepsilon^{\dot{D}\beta}_C (TQ)^{IC} (\lambda\tau Q)_{\dot{D}\beta B} + \frac{1}{3}\varepsilon^B_{\dot{A}\alpha} \varepsilon^{\dot{D}\beta}_B (TQ)^{IC} (Q\tau\lambda)_{C\dot{D}\beta}\end{aligned} \tag{C.8}$$

In the first term we just need to antisymmetrize the spin indices $\alpha\gamma$

$$\varepsilon^{B\beta}_{\dot{A}} \varepsilon^{\dot{C}\gamma}_B i\not{D}_{\beta\gamma} \lambda^I_{\dot{C}\alpha} + \varepsilon^{B\beta}_{\dot{A}} \varepsilon^{\dot{C}}_{B\alpha} i\not{D}^\gamma_\beta \lambda^I_{\dot{C}\gamma} \tag{C.9}$$

The first part is the usual translation, while the second part will go to the equations of motion.

The last three terms can be rearranged through the fundamental identity and recombined together. The total $QQ\lambda$ part is

$$\varepsilon^{B\beta}_{\dot{A}} \varepsilon^{C\dot{D}}_\alpha (TQ)^I_B (\lambda\tau Q)_{\beta\dot{D}C} + \varepsilon^{B\beta}_{\dot{A}} \varepsilon^{C\dot{D}}_\beta (TQ)^I_B k_{mn} (\lambda\tau Q)_{\alpha\dot{D}C} + \varepsilon^B_{\dot{A}\alpha} \varepsilon^{\dot{D}\beta}_C (TQ)^{IC} (Q\tau\lambda)_{B\dot{D}\beta} \tag{C.10}$$

. We expect to find the same gauge transformation as before:

$$\frac{1}{2}\varepsilon^{\dot{B}\alpha C}\varepsilon^{D}_{\dot{B}\alpha}\mu^{n}_{CD}k_{mn}T^{mI}_{J}\lambda^{J}_{\dot{A}} \tag{C.11}$$

We saw from the derivative term that the equations of motions instead should contain only the contraction $\varepsilon^{B\beta}_{\dot{A}}\varepsilon^{\dot{C}}_{B\alpha}$. The residual symmetric part, proportional to $\varepsilon^{(B}_{\dot{A}\alpha}\varepsilon^{C)}_{\dot{D}\beta}$ should cancel.

$$\begin{aligned}&\varepsilon^{B\beta}_{\dot{A}}\varepsilon^{C\dot{D}}_{\alpha}(TQ)^{I}_{B}(\lambda\tau Q)_{\beta\dot{D}C}+\varepsilon^{C\beta}_{\dot{A}}\varepsilon^{B\dot{D}}_{\alpha}(TQ)^{I}_{B}(\lambda\tau Q)_{\beta\dot{D}C}+\varepsilon^{B\beta}_{\dot{A}}\varepsilon^{C\dot{D}}_{\beta}(TQ)^{I}_{B}(\lambda\tau Q)_{\alpha\dot{D}C}+\\&\varepsilon^{C\beta}_{\dot{A}}\varepsilon^{B\dot{D}}_{\beta}(TQ)^{I}_{B}(\lambda\tau Q)_{\alpha\dot{D}C}-\varepsilon^{B}_{\dot{A}\alpha}\varepsilon^{\dot{D}\beta C}(TQ)^{I}_{C}(Q\tau\lambda)_{B\dot{D}\beta}-\varepsilon^{C}_{\dot{A}\alpha}\varepsilon^{\dot{D}\beta B}(TQ)^{I}_{C}(Q\tau\lambda)_{B\dot{D}\beta}\\&\qquad\qquad-\varepsilon^{\dot{B}\beta C}\varepsilon^{D}_{\dot{B}\beta}(Q\tau Q)_{CD}(T\lambda)^{I}_{\dot{A}\alpha}\end{aligned} \tag{C.12}$$

The first, second, third, fourth,fifth and sixth terms all come together and the expression simplifies to

$$-2\varepsilon^{\dot{B}\beta C}\varepsilon^{D}_{\beta\dot{B}}(TQ)^{I}_{D}(Q\tau\lambda)_{C\alpha\dot{A}}-\varepsilon^{\dot{B}\beta C}\varepsilon^{D}_{\dot{B}\beta}(Q\tau Q)_{CD}(T\lambda)^{I}_{\dot{A}\alpha} \tag{C.13}$$

This is zero by the fundamental identity. The remaining terms proportional to $\varepsilon^{B\beta}_{\dot{A}}\varepsilon^{\dot{C}}_{B\alpha}$ are

$$\frac{1}{2}\varepsilon^{\beta}_{\dot{A}B}\varepsilon^{B\dot{D}}_{\alpha}(TQ)^{I}_{C}(\lambda\tau Q)^{C}_{\beta\dot{D}}+\frac{1}{2}\varepsilon^{\beta}_{\dot{A}B}\varepsilon^{B\dot{D}}_{\beta}(TQ)^{I}_{C}(\lambda\tau Q)^{C}_{\alpha\dot{D}}-\frac{1}{2}\varepsilon_{\dot{A}\alpha B}\varepsilon^{\dot{D}\beta B}(TQ)^{IC}(Q\tau\lambda)_{C\dot{D}\beta} \tag{C.14}$$

It is straightforward to rearrange the spin indices and recombine everything to

$$\varepsilon^{\beta}_{\dot{A}B}\varepsilon^{B\dot{D}}_{\alpha}(TQ)^{I}_{C}k_{mn}(\lambda\tau Q)^{C}_{\beta\dot{D}} \tag{C.15}$$

The fermionic equations of motion are

$$\varepsilon^{B\beta}_{\dot{A}}\varepsilon^{\dot{C}}_{B\alpha}\left(i\not{D}^{\gamma}_{\beta}\lambda^{I}_{\dot{C}\gamma}-T^{mI}_{T}Q^{T}_{D}k_{mn}\tau^{n}_{KJ}Q^{JD}\lambda^{K}_{\beta\dot{C}}\right)=0 \tag{C.16}$$

Finally we want to look at the supersymmetry variations of the gauge fields

$$\delta^{2}\not{A}_{m\alpha\beta}=\varepsilon^{A\dot{B}}_{(\alpha}\varepsilon^{\dot{C}\gamma}_{A}(\lambda_{\dot{C}\gamma}\tau\lambda_{\beta)\dot{B}})+\varepsilon^{A\dot{B}}_{(\alpha}\varepsilon^{C\gamma}_{\dot{B}}(Q_{A}\tau i\not{D}_{\beta)\gamma}Q_{C})+\frac{1}{3}\varepsilon^{A\dot{B}}_{(\alpha}\varepsilon^{D}_{\beta)\dot{B}}k_{mn}(Q_{A}\tau^{n}T^{o}Q^{C})k_{op}(Q\tau^{p}Q)_{CD} \tag{C.17}$$

The first term is easy to discuss: the part of the ε bilinear which is symmetric in $\dot{B}\dot{C}$ drops out and the rest becomes

$$\frac{1}{2}\left(\varepsilon^{\dot{C}\gamma}_{A}\varepsilon^{A}_{\dot{C}(\alpha}\right)\left(\lambda^{\dot{B}}_{\beta)}\tau\lambda_{\dot{B}\gamma}\right)=\frac{1}{2}\left(\varepsilon^{\dot{C}\gamma}_{A}\varepsilon^{A}_{\dot{C}(\alpha}\right)k_{mn}\not{J}^{\lambda n}_{\beta)\gamma} \tag{C.18}$$

The second term written in full is

$$\varepsilon^{A\dot{B}}_{\alpha}\varepsilon^{C\gamma}_{\dot{B}}(Q_{A}\tau i\not{D}_{\beta\gamma}Q_{C})+\varepsilon^{A\dot{B}}_{\beta}\varepsilon^{C\gamma}_{\dot{B}}(Q_{A}\tau i\not{D}_{\alpha\gamma}Q)_{C} \tag{C.19}$$

The part symmetric in AC is the usual gauge transformation

$$-iD_{\alpha\beta}(\frac{1}{2}\varepsilon^{\dot{B}\alpha C}\varepsilon^{D}_{\dot{B}\alpha}\mu^{n}_{CD}k_{mn} \tag{C.20}$$

The antisymmetric part is

$$\frac{1}{2}\left(\varepsilon^{\dot{C}\gamma}_{A}\varepsilon^{A}_{\dot{C}(\alpha}\right)\left(Q_B\tau i\not{D}_{\beta)\gamma}Q^B\right) = \frac{1}{2}\left(\varepsilon^{\dot{C}\gamma}_{A}\varepsilon^{A}_{\dot{C}(\alpha}\right)k_{mn}\not{J}^{Qn}_{\beta)\gamma} \tag{C.21}$$

The third term is written in full as

$$\frac{1}{3}\varepsilon^{A\dot{B}}_{\alpha}\varepsilon^{D}_{\beta\dot{B}}k_{mn}(Q_A\tau^n T^o Q^C)k_{op}(Q\tau Q)_{CD} + \frac{1}{3}\varepsilon^{A\dot{B}}_{\beta}\varepsilon^{D}_{\alpha\dot{B}}k_{mn}(Q_A\tau^n T^o Q^C)k_{op}(Q\tau^p Q)_{CD} \tag{C.22}$$

The ε bilinears are actually antisymmetric in AD, hence we can simplify a bit

$$\frac{1}{3}\varepsilon^{\dot{B}}_{D\alpha}\varepsilon^{D}_{\beta\dot{B}}k_{mn}(Q_A\tau^n T^o Q^C)k_{op}(Q\tau^p Q)^{A}_{C} \tag{C.23}$$

The $Q^J_A\tau^n_{IJ}T^{oI}_S Q^{SC}$ multiplies a term symmetric in AC. From the relation $T^I_J\omega_{IK} = \tau_{JK}$ it is possible to see that the product of structure constants is made into a commutator by symmetrizing AC, and the term simplifies to something proportional to $\mu^p_{AC}\mu^{qAC}f_{pqm}$ which is zero by complete antisymmetry of the structure constants.

Hence we learn that

$$\delta^2 \not{A}_{m\alpha\beta} = \frac{1}{2}\left(\varepsilon^{\dot{C}\gamma}_{A}\varepsilon^{A}_{\dot{C}(\alpha}\right)k_{mn}\not{J}^{n}_{\beta)\gamma} \tag{C.24}$$

Comparison with the expected result gives the equations of motion for the gauge fields:

$$f_{m\alpha\beta} = \not{J}^{n}_{\alpha\beta} \tag{C.25}$$

Comparison with appendix B gives the normalization of the Chern Simons term: $\frac{k^{mn}}{2}A\wedge dA$, hence to get a canonical normalization we need to replace $k^{mn} \to \frac{k^{mn}}{2\pi}$.

The equations of motion for the fermions become

$$i\not{D}^{\gamma}_{\beta}\lambda^{I}_{\dot{C}\gamma} - 2\pi k_{mn}T^{mI}_{T}Q^{T}_{D}\tau^{n}_{KJ}Q^{JD}\lambda^{K}_{\beta\dot{C}}. \tag{C.26}$$

The Yukawa couplings must be $-\pi(\lambda^{\beta\dot{C}}\tau^m Q_D)k_{mn}(Q^D\tau^n\lambda_{\beta\dot{C}})$

Another useful normalization is the comparison with the $\mathcal{N} = 1$ formalism. Just set $\varepsilon^{A}_{\dot{B}\alpha} = \delta^{A}_{\dot{B}}\varepsilon_\alpha$ in the SUSY transformations to verify the values of the auxiliary fields: (we also

introduce the factor of 2π)

$$\begin{aligned}\delta Q^I_A &= \varepsilon^\alpha \lambda^I_{A\alpha} \\ \delta\lambda^I_{A\alpha} &= \varepsilon^\beta i\not{D}_{\alpha\beta} Q^I_A + \frac{2\pi}{3}\varepsilon_\alpha T^{mI}_J Q^{JC} Q^K_C \tau^n_{KT} Q^T_A k_{mn} \\ \delta\not{A}_{m\alpha\beta} &= 2\pi k_{mn}\varepsilon_{(\alpha}\lambda^{AI}_{\beta)}\tau^n_{IJ} Q^J_A\end{aligned} \tag{C.27}$$

We learn

$$F^I_A = -\frac{2\pi}{3} T^{mI}_J Q^{JC} Q^K_C \tau^n_{KT} Q^T_A k_{mn} \tag{C.28}$$

and

$$\chi_\alpha = 2\pi k_{mn}\lambda^{AI}_\alpha \tau^n_{IJ} Q^J_A \tag{C.29}$$

as expected.

D Relating $4d$ and $3d$ Expressions

Let us review again the $\mathcal{N}=4$ current multiplet in $3d$. For free hypermultiplets the gauge current is

$$J_{\beta\gamma} = \lambda^{\dot{B}}_\beta \tau \lambda_{\dot{B}\gamma} + Q_B \tau i\not{D}_{\beta\gamma} Q^B \tag{D.1}$$

The supersymmetry variation of the moment map defines the superpartner of the gauge current:

$$\delta\mu_{AB} = \varepsilon^{\dot{C}\alpha}_{(A}\lambda^I_{\dot{C}\alpha}\tau_{IJ} Q^J_{B)} = \varepsilon^{\dot{C}\alpha}_{(A} j_{B)\dot{C}\alpha} \quad j_{A\dot{B}\alpha} = \lambda^I_{\dot{B}\alpha}\tau_{IJ} Q^J_A \tag{D.2}$$

The supersymmetry variation of the current superpartner is then

$$\delta j_{A\dot{B}\alpha} = \varepsilon^{\dot{C}\beta}_A \lambda^J_{\dot{C}\beta}\lambda^I_{\dot{B}\alpha}\tau_{IJ} + Q^J_A \varepsilon^{C\beta}_{\dot{B}} i\not{D}_{\alpha\beta} Q^I_C \tau_{IJ} \tag{D.3}$$

We can separate various components by taking symmetric and antisymmetric parts in the bosons and fermions. The part symmetric in the two Q is

$$\frac{1}{2}\varepsilon^{C\beta}_{\dot{B}} i\not{D}_{\alpha\beta}\mu_{AC} \tag{D.4}$$

. The fermion bilinear which is a spacetime scalar is

$$\frac{1}{2}\varepsilon^{\dot{C}}_{A\alpha}\lambda^J_{\dot{C}\beta}\lambda^{I\beta}_{\dot{B}}\tau_{IJ} = \varepsilon^{\dot{C}}_{A\alpha} O_{\dot{C}\dot{B}} \quad O_{\dot{C}\dot{B}} = \frac{1}{2}\lambda^J_{\dot{C}\beta}\lambda^{I\beta}_{\dot{B}}\tau_{IJ} \tag{D.5}$$

and the remaining part is

$$\frac{1}{2}\varepsilon^\beta_{A\dot{B}}\lambda^J_{\dot{E}\beta}\lambda^{I\dot{E}}_\alpha\tau_{IJ} - \frac{1}{2}\epsilon^{DE} Q^J_D \varepsilon^\beta_{A\dot{B}} i\not{D}_{\alpha\beta} Q^I_E \tau_{IJ} = -\frac{1}{2}\varepsilon^\beta_{A\dot{B}} \not{J}_{\alpha\beta} \tag{D.6}$$

In the paper we use boundary conditions of the form $Y_{AB} = c\mu_{AB}$. This equation makes sense because both Y^{AB} and the moment map are the leading components of two identical $OSp(4|4)$ supermultiplets. As a result, under 3d $\mathcal{N} = 4$ supersymmetry variation one gets a multiplet of boundary conditions equating corresponding members of the supermultiplets. Indeed, the fields of the $\mathcal{N} = 4$ gauge multiplet in four dimensions decompose under the $3d$ $\mathcal{N} = 4$ supergroup into two multiplets which are identical in quantum numbers to the current supermultiplet or to the mirror current supermultiplet respectively (Y has spin $(1, 0)$ and X has spin $(0, 1)$ under the R-symmetry group). The precise decomposition depends on the value of ψ. The supersymmetry variation

$$\delta Y^a = i\bar{\varepsilon}\Gamma^a\Psi = -\frac{i}{2}\epsilon^{abc}\bar{\varepsilon}\Gamma_{bc}\Gamma^3 B_1\Psi \tag{D.7}$$

As the generators Γ_{bc} act on V_8 only, the supersymmetry variation involves the projection of Ψ on a specific vector in V_2. If we use the decomposition

$$\Psi = B_1\varepsilon_0 \otimes \Psi_1 + B_2\varepsilon_0 \otimes \Psi_2 \tag{D.8}$$

we can rewrite the supersymmetry variation of Y

$$\delta Y^a = -\frac{i}{2}\epsilon^{abc}(\bar{\varepsilon}\Gamma^3 B_0\varepsilon_0)\Gamma_{bc}\Psi_1 \tag{D.9}$$

Now that everything happens in V_8 we can reintroduce the $SO(2, 1)\times SO(3)\times SO(3)$ indices:

$$\delta Y_{AB} = \varepsilon_{(A}^{\dot{C}\alpha}\Psi_{1B)\dot{C}\alpha} \tag{D.10}$$

and discover the boundary condition $\Psi_{1B\dot{C}\alpha} = cj_{B\dot{C}\alpha}$. For the next step, we need

$$\delta\Psi_1 = \bar{\varepsilon}_0 B_2\Psi = \frac{1}{2}\bar{\varepsilon}_0 B_2\Gamma^{IJ}F_{IJ}\varepsilon. \tag{D.11}$$

There are several contributions. The gamma matrix bilinears can be rewritten as various B_i times generators of $SO(2, 1)\times SO(3)\times SO(3)$. These B_i matrices are sandwiched between $\bar{\varepsilon}_0$ and $\varepsilon = \varepsilon_0\otimes\eta$, and we can compute the inner products in V_2 right away: if the matrix is B_2 the term drops out, as $\bar{\varepsilon}_0\varepsilon = 0$, if the matrix is B_1 the result is ψ independent, if it is 1 the result is proportional to $-\sin\psi$, if it is B_0 it is proportional to $-\cos\psi$. The D_3Y^p has B_2 and corresponds to the derivative of the moment map in the δj. The gauge field strengths come as $\frac{1}{2}F_{\mu\nu}\epsilon^{\mu\nu\rho}\sin\psi + F^{\rho 3}\cos\psi$, and should be equal to cJ^ρ. Physically, we know that

$$F^{\rho 3} = \frac{1}{2}J^\rho - \frac{1}{2}F_{\mu\nu}\epsilon^{\mu\nu\rho}\tan\psi, \tag{D.12}$$

where the second piece on the right hand side is the current induced by the Chern Simons term a the boundary or by the theta-angle. The factor of two in front of the current is due to the slightly non-standard normalization of the gauge field kinetic term. Hence we learn that $c = \frac{1}{2}\cos\psi$ and $\frac{g^2\theta_{YM}}{8\pi^2} = \frac{\sin\psi}{\cos\psi}$.

Finally, there are terms as $D_3 X_{\dot{A}\dot{B}}$ and $-\sin\psi[X,X]_{\dot{A}\dot{B}}$ whose sum equals $cO_{\dot{A}\dot{B}}$.

Along similar lines we could compare the Yukawa couplings computed with the $3d$ formalism to the conventional $\mathcal{N}=4$ ones. If we plug

$$\Psi = B_1\varepsilon_0 \otimes \Psi_1 + B_2\varepsilon_0 \otimes \Psi_2 \tag{D.13}$$

in the fermion kinetic terms in $\mathcal{N}=4$ super Yang-Mills we get an expression in V_8

$$-\frac{i}{e^2}\overline{\Psi}\Gamma^\mu D_\mu \Psi = -\frac{i}{2e^2}\epsilon^{\mu\nu\rho}\sum_i \overline{\Psi}_i \Gamma_{\nu\rho} D_\mu \Psi_i. \tag{D.14}$$

we can reintroduce the $SO(2,1)\otimes SO(3)\otimes SO(3)$ indices

$$\frac{1}{e^2}\sum_i \Psi_i^{A\dot{B}\alpha} \not{D}_\alpha^\beta \Psi_{iA\dot{B}\beta} \tag{D.15}$$

The X Yukawa couplings in $\mathcal{N}=4$ super Yang-Mills are

$$-\frac{i}{e^2}\overline{\Psi}\Gamma^a[X^a,\Psi] = \frac{i}{2e^2}\epsilon^{abc}\overline{\Psi}\Gamma_{bc}\Gamma^3 B_1[X^a,\Psi] \tag{D.16}$$

If we plug in

$$\Psi = B_1\varepsilon_0 \otimes \Psi_1 + B_2\varepsilon_0 \otimes \Psi_2 \tag{D.17}$$

we get

$$\frac{i}{2e^2}\epsilon^{abc}\left(-\cos\psi\overline{\Psi}_1\Gamma_{bc}[X^a,\Psi_1] + \sin\psi\overline{\Psi}_1\Gamma_{bc}[X^a,\Psi_2] + \sin\psi\overline{\Psi}_2\Gamma_{bc}[X^a,\Psi_1] + \cos\psi\overline{\Psi}_2\Gamma_{bc}[X^a,\Psi_2]\right) \tag{D.18}$$

Now that everything happens in V_8 we can reintroduce the $SO(2,1)\otimes SO(3)\otimes SO(3)$ indices:

$$-\frac{1}{e^2}\left(-\cos\psi\Psi^{A\alpha}_{1\dot{B}}[X^{\dot{B}\dot{C}},\Psi_{1A\dot{C}\alpha}] + 2\sin\psi\Psi^{A\alpha}_{1\dot{B}}[X^{\dot{B}\dot{C}},\Psi_{2A\dot{C}\alpha}] + \cos\psi\Psi^{A\alpha}_{2\dot{B}}[X^{\dot{B}\dot{C}},\Psi_{2A\dot{C}\alpha}]\right) \tag{D.19}$$

This agrees with the computation in the text.

References

[1] D. Bak, M. Gutperle and S. Hirano, *A Dilatonic Deformation of AdS(5) and its Field Theory Dual*, *JHEP* **05** (2003) 072 [hep-th/0304129].

[2] A. B. Clark, D. Z. Freedman, A. Karch and M. Schnabl, *The Dual of Janus ((<:) ↔ (:>)) an Interface CFT*, *Phys. Rev.* **D71** (2005) 066003 [hep-th/0407073].

[3] A. Clark and A. Karch, *Super Janus*, *JHEP* **10** (2005) 094 [hep-th/0506265].

[4] E. D'Hoker, J. Estes and M. Gutperle, *Exact Half-BPS Type IIB Interface Solutions I: Local Solution and Supersymmetric Janus*, *JHEP* **06** (2007) 021 [arXiv:0705.0022 [hep-th]].

[5] E. D'Hoker, J. Estes and M. Gutperle, *Interface Yang-Mills, Supersymmetry, and Janus*, hep-th/0603013.

[6] H.-C. Kao and K.-M. Lee, *Selfdual Chern-Simons systems with an N=3 extended supersymmetry*, *Phys. Rev.* **D46** (1992) 4691–4697 [hep-th/9205115].

[7] H.-C. Kao, K.-M. Lee and T. Lee, *The Chern-Simons Coefficient in Supersymmetric Yang-Mills Chern-Simons Theories*, *Phys. Lett.* **B373** (1996) 94–99 [hep-th/9506170].

[8] A. Kapustin and M. J. Strassler, *On Mirror Symmetry in Three Dimensional Abelian Gauge Theories*, *JHEP* **04** (1999) 021 [hep-th/9902033].

[9] D. Gaiotto and X. Yin, *Notes on Superconformal Chern-Simons-Matter Theories*, *JHEP* **08** (2007) 056 [0704.3740].

[10] J. H. Schwarz, *Superconformal Chern-Simons Theories*, *JHEP* **11** (2004) 078 [hep-th/0411077].

[11] J. Bagger and N. Lambert, *Gauge Symmetry and Supersymmetry of Multiple M2-Branes*, *Phys. Rev.* **D77** (2008) 065008 [arXiv:0711.0955 [hep-th]].

[12] L. Brink, J. H. Schwarz and J. Scherk, *Supersymmetric Yang-Mills Theories*, *Nucl. Phys.* **B121** (1977) 77.

[13] S. Deser, R. Jackiw and S. Templeton, *Topologically Massive Gauge Theories*, *Ann. Phys.* **140** (1982) 372–411.

[14] S. J. Gates, M. T. Grisaru, M. Rocek and W. Siegel, *Superspace, or One Thousand and One Lessons in Supersymmetry*, *Front. Phys.* **58** (1983) 1–548 [hep-th/0108200].